

Mathematical Methods

Physics 605

Lectures by Prof. Tigran Sedrakyan

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Fall 2025

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Introduction

These notes were compiled by **Siyi Zhang** for **Physics 605: Mathematical Methods**, taught by **Prof. Tigran Sedrakyan** at the **University of Massachusetts Amherst** in **Fall 2025**.

The course develops the mathematical language used in classical and quantum physics. Complex analysis leads to contour integration and dispersion relations. Function spaces and differential operators provide the framework for eigenfunction expansions, Green's functions, and partial differential equations.

Course content

The main parts follow the order of topics in the syllabus:

1. **Complex analysis.** Complex numbers, holomorphic functions, Cauchy's theorem and formula, series, singularities, and residue calculus.
2. **Vector and function spaces.** Inner products, orthogonal expansions, Hilbert spaces, and distributions.
3. **Differential operators.** Adjoint operators, boundary conditions, Sturm–Liouville theory, and discrete and continuous spectra.
4. **Linear response.** Green's functions, solvability, causality, damped oscillators, and dispersion relations.
5. **Fourier analysis.** Transform conventions, spectral representations, multidimensional transforms, and radial transforms.
6. **Special functions.** Legendre polynomials, Bessel functions, and spherical harmonics.
7. **Partial differential equations.** Classification, characteristics, conservation laws, diffusion, waves, and boundary problems.
8. **Electrostatic and magnetostatic applications.** Poisson and Laplace equations, multipole expansions, and vector potentials.

The supplied lectures on groups and representations form a final supplementary part, including angular momentum and finite-dimensional Lorentz representations.

Notation

The complex inner product is conjugate-linear in its first argument. Spatial Fourier transforms use $e^{-i\mathbf{k}\cdot\mathbf{x}}$ in the forward transform; temporal transforms use $e^{i\omega t}$. Each convention is stated where first used. Boundary and initial conditions are part of the specification of every differential problem.

References and further reading

Syllabus reading list

The instructor provides detailed lecture notes and does not follow a single textbook. The first two books below are especially recommended in the syllabus.

1. Michael Stone and Paul Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students*. An advanced graduate treatment, particularly useful for theoretical physics. The syllabus links to the authors' pre-publication version: https://goldbart.gatech.edu/PG_MS_MfP.htm.
2. Hans J. Weber and George B. Arfken, *Essential Mathematical Methods for Physicists*. A basic graduate treatment of mathematical methods.
3. J. D. Jackson, *Classical Electrodynamics*. A reference for applications in electrostatics and magnetostatics.
4. Lecture notes made available by Prof. Prokofiev and Prof. Davidovitch. The syllabus lists <https://people.umass.edu/~prokofev/P605/605.htm>.

Additional supplied materials

The source collection also includes a spherical-harmonics handout, notes on conservation and dissipation, a first-order PDE reading, wave-equation slides, and Y. D. Chong's chapter on Green's functions from *Complex Methods for the Sciences* (2021).

Further references cited in the spherical-harmonics handout include:

- G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists*, 7th edition.
- Susan M. Lea, *Mathematics for Physicists*.
- Wu-Ki Tung, *Group Theory in Physics*.
- M. Chaichian and R. Hagedorn, *Symmetries in Quantum Mechanics: From Angular Momentum to Supersymmetry*.
- D. S. Mitrinović, *Analytic Inequalities*.
- Francis E. Low, *Classical Field Theory: Electromagnetism and Gravitation*.

A source-to-topic index is provided at the end of these notes.

Part I

Complex Analysis

Complex
analysis

1 Complex numbers and elementary functions

1.1 Complex arithmetic as arithmetic of ordered pairs

A complex number is an ordered pair of real numbers together with a specified addition and multiplication. Addition is the usual vector addition in two dimensions:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Multiplication is defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2). \quad (1.1)$$

The result is another ordered pair. This operation is different from a real scalar product, whose output is a real number.

Identify $(x, 0)$ with the real number x , and write

$$1 = (1, 0), \quad i = (0, 1).$$

Then every ordered pair can be written as

$$(x, y) = x(1, 0) + y(0, 1) = x + iy.$$

The multiplication rule immediately gives

$$1 \cdot 1 = 1, \quad 1 \cdot i = i, \quad i^2 = (0, 1)(0, 1) = (-1, 0) = -1.$$

Thus the arithmetic rules are

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \quad (1.2)$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \quad (1.3)$$

We denote the set of complex numbers by $\mathbb{C}$.

The enlargement from $\mathbb{R}$ to $\mathbb{C}$ allows equations such as $z^2 + 1 = 0$ to have solutions:

$$z^2 + 1 = (z - i)(z + i).$$

More generally, every nonconstant polynomial with complex coefficients factors into linear factors over $\mathbb{C}$. If its distinct roots are $z_1, \dots, z_k$ and their multiplicities are $n_1, \dots, n_k$, then

$$P(z) = a_n \prod_{j=1}^k (z - z_j)^{n_j}, \quad \sum_{j=1}^k n_j = n. \quad (1.4)$$

This is the fundamental theorem of algebra. A proof using complex analysis will follow in Section 3.7.

1.2 Conjugation, modulus, and division

For $z = x + iy$, its complex conjugate is

$$\bar{z} = x - iy.$$

Conjugation reflects the point representing z across the real axis. It obeys

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\bar{z}} = z.$$

The real and imaginary parts can be recovered by

$$x = \operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad y = \operatorname{Im} z = \frac{z - \bar{z}}{2i}. \quad (1.5)$$

The modulus is the Euclidean distance from the origin:

$$|z| = \sqrt{x^2 + y^2}, \quad z\bar{z} = x^2 + y^2 = |z|^2.$$

For $z \neq 0$, multiplication by the conjugate gives its inverse:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}.$$

In particular,

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}, \quad z_2 \neq 0.$$

A useful connection with vector algebra is

$$\bar{z}_1 z_2 = \underbrace{x_1 x_2 + y_1 y_2}_{\text{scalar product}} + i \underbrace{(x_1 y_2 - y_1 x_2)}_{\text{oriented area}}.$$

Thus the real and imaginary parts of one complex product encode the scalar product and the two-dimensional determinant.

1.3 Polar representation and Euler's formula

For a nonzero complex number, write

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r = |z| > 0.$$

Then

$$z = r(\cos \theta + i \sin \theta).$$

The angle θ is an **argument** of z . It is determined only modulo 2π : all angles $\theta + 2\pi n$, $n \in \mathbb{Z}$, describe the same point. The number zero has no argument.

Multiplying two polar expressions and using the angle addition identities gives

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]. \end{aligned}$$

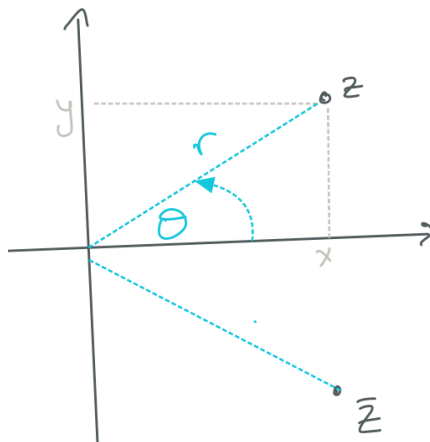


Figure 1: Polar representation and complex conjugation.

Moduli multiply and arguments add. Multiplication by a fixed nonzero complex number is therefore a rotation followed by a dilation.

To establish the exponential notation precisely, define

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The ratio of successive absolute values is $|z|/(n+1)$, which tends to zero, so the series converges for every $z \in \mathbb{C}$. Substituting $z = i\theta$ and separating even and odd powers gives

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right). \end{aligned}$$

The two series are the cosine and sine series, so

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (1.6)$$

Consequently,

$$z = re^{i\theta}, \quad \bar{z} = re^{-i\theta}, \quad (1.7)$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}, \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}. \quad (1.8)$$

For an integer n , this gives de Moivre's formula,

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta).$$

Taking the imaginary part of $e^{i(\theta_1 + \theta_2)} = e^{i\theta_1} e^{i\theta_2}$, for example, gives

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2.$$

This is an efficient way to derive trigonometric identities.

Since $z = re^{i\theta}$, the possible logarithms of $z \neq 0$ are

$$\log z = \log r + i(\theta + 2\pi n), \quad n \in \mathbb{Z}.$$

A single-valued logarithm requires a choice of the argument. On a chosen local branch, the argument is the imaginary part of $\log z$. Global branch choices will be studied in Section 7.

1.4 Complex trigonometric and hyperbolic functions

Define the complex sine and cosine by their convergent power series,

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

Separating even and odd powers in e^{iz} and e^{-iz} yields

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \quad (1.9)$$

Similarly,

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}. \quad (1.10)$$

All four functions are real on the real axis, but their values are usually complex away from it.

Exercise 1.1 (Trigonometric identities for complex arguments). Prove that

$$\cos^2 z + \sin^2 z = 1, \quad \cosh^2 z - \sinh^2 z = 1, \quad z \in \mathbb{C}.$$

Proof. Put $A = e^{iz}$ and $B = e^{-iz}$. Then $AB = 1$, and

$$\cos^2 z + \sin^2 z = \frac{(A+B)^2 - (A-B)^2}{4} = AB = 1.$$

The same calculation with $A = e^z$ and $B = e^{-z}$ proves the hyperbolic identity. $\square$

2 Holomorphic functions and the Cauchy–Riemann equations

2.1 Complex differentiability and analyticity

A domain is an open connected subset of $\mathbb{C}$. A domain is **simply connected** if every closed curve in it can be continuously contracted to a point while staying in the domain. A disk is simply connected; a punctured disk and an annulus are not.

Definition 2.1. The complex derivative of f at z_0 is

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}, \quad h \in \mathbb{C}, \quad (2.1)$$

when the limit exists. A function is **holomorphic** on an open set if it is complex differentiable at every point of that set. It is **analytic** if it has a convergent local power series around every point. It is **entire** if it is holomorphic on all of $\mathbb{C}$.

A central theorem of complex analysis is that holomorphic and analytic functions are the same class. In particular, differentiability on an open neighborhood implies the existence of derivatives of every order there. Differentiability at only one isolated point is not sufficient for this conclusion.

Polynomials, e^z , $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ are entire. A function can also be holomorphic on a domain that is not simply connected: $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Simple connectedness will be important for global integration statements, rather than for the definition of a local derivative.

The complex result is much stronger than its real-variable analogue. For example, the real function $f(x) = |x|^5$ has four continuous derivatives everywhere, but

$$f^{(5)}(x) = \begin{cases} 120, & x > 0, \\ -120, & x < 0, \end{cases}$$

and the fifth derivative does not exist at zero. Such a loss of higher derivatives cannot occur for a function holomorphic in a neighborhood.

2.2 Deriving the Cauchy–Riemann equations

Write

$$f(z) = f(x, y) = u(x, y) + iv(x, y), \quad z = x + iy,$$

where u and v are real-valued. The difference quotient in (2.1) must approach the same limit for every direction of h . First take a horizontal increment, $h = \Delta x$. Then

$$f'(z) = u_x + iv_x.$$

Next take a vertical increment, $h = i\Delta y$. This gives

$$f'(z) = \frac{u_y + iv_y}{i} = v_y - iu_y.$$

Comparing real and imaginary parts yields the **Cauchy–Riemann equations**:

$$u_x = v_y, \quad u_y = -v_x. \quad (2.2)$$

These equations are necessary for complex differentiability. If u and v have continuous first partial derivatives in a neighborhood and satisfy (2.2), they are also sufficient. Indeed, real differentiability gives

$$\begin{aligned} \Delta f &= u_x \Delta x + u_y \Delta y + i(v_x \Delta x + v_y \Delta y) + o(|\Delta z|) \\ &= (u_x + iv_x)(\Delta x + i\Delta y) + o(|\Delta z|), \end{aligned}$$

where the second equality uses the Cauchy–Riemann equations. Dividing by Δz proves that the derivative is $u_x + iv_x$ independently of the direction of approach.

The real Jacobian has the special form

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a = u_x, \quad b = v_x. \quad (2.3)$$

When $f'(z) = a + ib \neq 0$, this is a dilation by $|f'(z)|$ and a rotation by $\arg f'(z)$. Thus a holomorphic map preserves oriented angles locally wherever its derivative is nonzero.

Example 2.2 (A real-valued holomorphic function). Suppose $f = u + iv$ is holomorphic on a connected domain and all its values are real. Then $v = 0$, so $v_x = v_y = 0$. The Cauchy–Riemann equations imply $u_x = u_y = 0$. Therefore u , and hence f , is constant. The hypothesis concerns real values throughout an open domain, not merely values along the real axis.

2.3 The variables z and $\bar{z}$

Equations (1.5) motivate the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \quad (2.4)$$

They are called Wirtinger derivatives. They act on functions of the two real coordinates; the notation does not require z and $\bar{z}$ to be independently variable physical coordinates. The inverse relations are

$$\partial_x = \partial_z + \partial_{\bar{z}}, \quad \partial_y = i(\partial_z - \partial_{\bar{z}}).$$

For example,

$$\partial_z z = 1, \quad \partial_z \bar{z} = 0, \quad \partial_{\bar{z}} z = 0, \quad \partial_{\bar{z}} \bar{z} = 1.$$

For $f = u + iv$, direct substitution gives

$$\partial_{\bar{z}} f = \frac{1}{2} [(u_x - v_y) + i(v_x + u_y)].$$

Consequently, for a continuously differentiable function,

$$f \text{ is holomorphic} \iff \partial_{\bar{z}} f = 0, \quad f'(z) = \partial_z f. \quad (2.5)$$

This compact formulation is useful when complex coordinates appear in Hamiltonian mechanics and quantum mechanics. For instance, $|z|^2 = z\bar{z}$ has $\partial_{\bar{z}}|z|^2 = z$, so it is not holomorphic on any open set.

2.4 Analytic continuation from real data

Suppose we seek a holomorphic function near the real axis with

$$f(x, 0) = x^2.$$

The candidate $f(z) = z^2$ works, since

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Thus

$$u = x^2 - y^2, \quad v = 2xy,$$

and the Cauchy–Riemann equations hold:

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x.$$

The imaginary part must be zero on the real axis, fixing any additive imaginary constant. The desired continuation is therefore

$$f(z) = z^2.$$

One can also construct it through harmonic functions. A holomorphic real part must be harmonic, and the prescribed data give $u(x, 0) = x^2$. Since $v(x, 0) = 0$, the Cauchy–Riemann equations also give $u_y(x, 0) = -v_x(x, 0) = 0$. The harmonic extension $u = x^2 - y^2$ has these data, and integrating $v_y = u_x = 2x$ gives $v = 2xy + C(x)$. The other Cauchy–Riemann equation implies $C'(x) = 0$; $v(x, 0) = 0$ then gives $C = 0$.

The uniqueness principle is that two holomorphic functions on a connected domain which agree on a set with an accumulation point inside that domain agree everywhere. To see the local reason, consider their difference h . If its first nonzero Taylor coefficient at z_0 has order m , then

$$h(z) = (z - z_0)^m [a_m + O(z - z_0)], \quad a_m \neq 0.$$

Its zero at z_0 is isolated. Accumulating zeros therefore force every Taylor coefficient to vanish, and the vanishing extends throughout the connected domain. This explains why replacing a real variable by z can produce a unique continuation, when a holomorphic continuation exists.

2.5 Harmonic functions and planar vector fields

The Cauchy–Riemann equations imply that both u and v satisfy Laplace’s equation. Differentiating gives

$$\begin{aligned} u_{xx} + u_{yy} &= v_{yx} - v_{xy} = 0, \\ v_{xx} + v_{yy} &= -u_{yx} + u_{xy} = 0. \end{aligned}$$

Thus

$$\nabla^2 u = 0, \quad \nabla^2 v = 0. \tag{2.6}$$

The pair u, v are called harmonic conjugates. In Wirtinger notation,

$$\nabla^2 = \partial_x^2 + \partial_y^2 = 4\partial_z \partial_{\bar{z}}.$$

Two vector fields associated with $f = u + iv$ are

$$\mathbf{F}_1 = (u, -v), \quad \mathbf{F}_2 = (v, u).$$

Their divergences are

$$\nabla \cdot \mathbf{F}_1 = u_x - v_y = 0, \quad \nabla \cdot \mathbf{F}_2 = v_x + u_y = 0,$$

and their two-dimensional curls are

$$(\nabla \times \mathbf{F}_1)_z = -v_x - u_y = 0, \quad (\nabla \times \mathbf{F}_2)_z = u_x - v_y = 0.$$

They are both divergence-free and locally conservative. On a simply connected domain they have globally defined scalar potentials. These connections explain the role of complex functions in two-dimensional electrostatics and incompressible potential flow.

3 Contour integration and Cauchy's formulas

3.1 Contour integrals and their orientation

Let a piecewise smooth curve Γ be parametrized by $z = z(t)$, $a \leq t \leq b$. Its complex line integral is

$$\int_{\Gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt. \quad (3.1)$$

Reversing the orientation changes the sign. A positively oriented simple closed curve is traversed counterclockwise, leaving its interior on the left. For $f = u + iv$ and $dz = dx + idy$,

$$\int_{\Gamma} f(z) dz = \int_{\Gamma} (u dx - v dy) + i \int_{\Gamma} (v dx + u dy). \quad (3.2)$$

A basic estimate used repeatedly below is

$$\left| \int_{\Gamma} f(z) dz \right| \leq \int_{\Gamma} |f(z)| |dz| \leq \left(\sup_{\Gamma} |f| \right) \text{length}(\Gamma). \quad (3.3)$$

3.2 Cauchy's integral theorem

Theorem 3.1 (Cauchy's theorem). *If f is holomorphic on a neighborhood of a simple closed curve Γ and its interior, then*

$$\oint_{\Gamma} f(z) dz = 0. \quad (3.4)$$

More generally, the same result holds for closed curves in a simply connected domain on which f is holomorphic.

Proof using Green's theorem. Assume first that the real partial derivatives are continuous. With D the region enclosed by Γ , Green's theorem and (3.2) give

$$\begin{aligned} \oint_{\Gamma} f(z) dz &= \iint_D (-v_x - u_y) dx dy + i \iint_D (u_x - v_y) dx dy \\ &= 0, \end{aligned}$$

because each integrand vanishes by the Cauchy–Riemann equations. Cauchy's theorem also holds without separately assuming continuity of the derivative. $\square$

The domain condition is essential. Although $1/z$ is holomorphic away from zero,

$$\oint_{|z|=r} \frac{dz}{z} = \int_0^{2\pi} \frac{ir e^{i\theta}}{r e^{i\theta}} d\theta = 2\pi i.$$

The singularity at zero lies inside the contour, so Cauchy's theorem does not apply to its full interior.

3.3 Path independence and contour deformation

If two paths Γ_1, Γ_2 run from a to b and can be deformed into one another without crossing a singularity or leaving the holomorphic domain, then the closed curve consisting of Γ_1 followed by $-\Gamma_2$ has zero integral. Hence

$$\int_{\Gamma_1} f(z) dz - \int_{\Gamma_2} f(z) dz = 0.$$

In a simply connected domain, the integral between two points is therefore independent of the chosen path. Fixing a defines a primitive

$$F(z) = \int_a^z f(\zeta) d\zeta, \quad F'(z) = f(z).$$

Similarly, two closed contours have the same integral if one can be deformed into the other through a region where the integrand is holomorphic. This allows a complicated contour to be replaced by small circles around its singularities.

3.4 Cauchy's integral formula and its proof

Theorem 3.2 (Cauchy's integral formula). *Let f be holomorphic on a neighborhood of a positively oriented simple closed curve Γ and its interior. If z_0 is inside Γ , then*

$$f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} dz. \quad (3.5)$$

Proof. The quotient $f(z)/(z - z_0)$ is holomorphic away from z_0 . Remove a small disk $|z - z_0| < \varepsilon$ and apply Cauchy's theorem to the remaining region. Its outer boundary is counterclockwise and its inner boundary is clockwise. If C_ε denotes the small circle with counterclockwise orientation, the result is

$$\oint_{\Gamma} \frac{f(z)}{z - z_0} dz - \oint_{C_\varepsilon} \frac{f(z)}{z - z_0} dz = 0.$$

Equivalently, one may connect the two boundaries by a thin cut; the integrals along its two sides cancel because their orientations are opposite.

Parametrize the small circle by

$$z = z_0 + \varepsilon e^{i\theta}, \quad dz = i\varepsilon e^{i\theta} d\theta, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\oint_{C_\varepsilon} \frac{f(z)}{z - z_0} dz = i \int_0^{2\pi} f(z_0 + \varepsilon e^{i\theta}) d\theta.$$

Continuity at z_0 makes the last integrand tend uniformly to $f(z_0)$ as $\varepsilon \rightarrow 0$. The limit is $2\pi i f(z_0)$, which proves the formula. $\square$

Cauchy's formula says that the values of a holomorphic function on an enclosing contour determine its value at every interior point. For a harmonic potential u , one can form a holomorphic function $u + iv$ locally by finding a harmonic conjugate v . If its complex boundary values are known, Cauchy's formula reconstructs the interior. For a Dirichlet problem in which only u is prescribed on the boundary, the conjugate boundary values must also be determined, or one may use an equivalent harmonic boundary formula such as the Poisson integral.

3.5 Derivatives from values on an enclosing contour

Take z_0 and $z_0 + h$ inside the same fixed contour. Subtracting their Cauchy formulas gives

$$\begin{aligned} f(z_0 + h) - f(z_0) &= \frac{1}{2\pi i} \oint_{\Gamma} f(z) \left[\frac{1}{z - z_0 - h} - \frac{1}{z - z_0} \right] dz \\ &= \frac{h}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0 - h)(z - z_0)} dz. \end{aligned}$$

The contour stays a positive distance from z_0 . We can therefore divide by h and pass the limit through the integral:

$$f'(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^2} dz. \quad (3.6)$$

Differentiating once more with respect to z_0 gives

$$f''(z_0) = \frac{2}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^3} dz.$$

Repeating the argument yields the general result

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n = 0, 1, 2, \dots \quad (3.7)$$

Every one of these integrals exists. Thus a holomorphic function has all higher derivatives. A derivative describes local variation, yet (3.7) calculates it entirely from values away from the evaluation point.

3.6 Taylor series and convergence radii

Suppose f is holomorphic on and inside the circle $|\zeta - z_0| = R$. For $|z - z_0| < R$,

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0} \frac{1}{1 - (z - z_0)/(\zeta - z_0)} \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.\end{aligned}$$

The geometric series converges uniformly on the contour for fixed $|z - z_0| < R$. Substitution into Cauchy's formula gives

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (3.8)$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \frac{f^{(n)}(z_0)}{n!}. \quad (3.9)$$

This proves the local power-series representation, and hence the equivalence of holomorphicity and analyticity.

The Taylor series about z_0 converges throughout the largest disk centered at z_0 to which that holomorphic function extends. For the meromorphic examples below, its radius is the distance to the nearest nonremovable singularity. A series can fail on the real axis because of a singularity away from the real axis.

Example 3.3 (A geometric series). For $|z| < 1$,

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots.$$

The pole at $z = -1$ is one unit from the center, so the radius is one. The real function $1/(1+x)$ is smooth at $x = 1$, but its Taylor series about zero still does not converge there. The radius describes a disk in the complex plane, not just the interval where the real function is smooth.

Example 3.4 (Complex poles restricting a real expansion). The function $1/(1+x^2)$ is smooth for every real x . Its complex continuation has poles at $z = \pm i$, so

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \quad |z| < 1.$$

There is no singularity at $x = \pm 1$ on the real axis, but these points lie on the boundary of the disk of convergence.

Nonremovable obstructions can be isolated singularities, branch points requiring cuts for a chosen single-valued branch, or accumulation points of singularities. For example, $1/\sin(1/z)$ has poles at $z = 1/(n\pi)$ for nonzero integers n ; these accumulate at zero. Thus zero is not an isolated singularity of that function. Isolated singularities are treated next, and branches in Section 7.

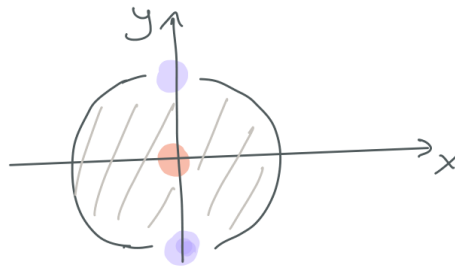


Figure 2: The nearest complex singularities determine the Taylor disk.

3.7 Cauchy estimates, Liouville's theorem, and algebra

On a circle $|z - z_0| = R$ where $|f(z)| \leq M_R$, the derivative formula and the contour-length estimate give

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} (2\pi R) = \frac{n! M_R}{R^n}. \quad (3.10)$$

Theorem 3.5 (Liouville's theorem). *A bounded entire function is constant.*

Proof. If $|f(z)| \leq M$ throughout the plane, we may use any radius R in (3.10). For $n = 1$,

$$|f'(z_0)| \leq \frac{M}{R}.$$

Letting $R \rightarrow \infty$ gives $f'(z_0) = 0$. The point z_0 was arbitrary, so f is constant. $\square$

Theorem 3.6 (Fundamental theorem of algebra). *Every polynomial of degree $n \geq 1$ with complex coefficients has exactly n complex roots counted with multiplicity.*

Proof. Let $P(z) = a_n z^n + \cdots + a_0$ with $a_n \neq 0$. Suppose it has no root. Then $1/P(z)$ is entire. For sufficiently large $|z|$, the leading term dominates:

$$P(z) = a_n z^n \left[1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_0}{a_n z^n} \right], \quad |P(z)| \geq \frac{1}{2} |a_n| |z|^n.$$

Hence $1/P(z) \rightarrow 0$ at infinity and is bounded outside a large disk. Inside that disk it is continuous and therefore bounded. Liouville's theorem says that $1/P$ is constant, which contradicts $\deg P \geq 1$. Thus P has at least one root z_1 .

Polynomial division gives $P(z) = (z - z_1)P_{n-1}(z)$. If $n - 1 \geq 1$, the same argument applies to P_{n-1} . Repeating until the remaining factor is constant gives

$$P(z) = a_n (z - z_1) \cdots (z - z_n).$$

Repeated roots are repeated factors, giving (1.4). $\square$

4 Laurent series and isolated singularities

4.1 An annular version of Cauchy's formula

A Taylor series contains nonnegative powers because its center is a point of analyticity. If the center is excluded, negative powers may be needed. Suppose f is holomorphic in the annulus

$$r < |z - z_0| < R.$$

Choose radii $r < \rho_- < |z - z_0| < \rho_+ < R$ and let C_- and C_+ be the corresponding circles, both oriented counterclockwise. Cauchy's formula applied to the annular region gives

$$f(z) = \frac{1}{2\pi i} \oint_{C_+} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{C_-} \frac{f(\zeta)}{\zeta - z} d\zeta. \quad (4.1)$$

The minus sign is the inner-boundary orientation. Connecting the two circles by opposite copies of the same radial path makes a simple closed boundary; the connecting contributions cancel.

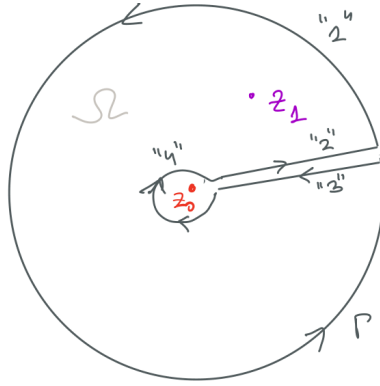


Figure 3: The annular contour used to derive the Laurent expansion.

On the outer circle, $|z - z_0| < |\zeta - z_0|$, so

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

On the inner circle the opposite inequality holds, so instead use

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0} \frac{1}{1 - (\zeta - z_0)/(z - z_0)} = -\sum_{m=0}^{\infty} \frac{(\zeta - z_0)^m}{(z - z_0)^{m+1}}.$$

Substituting both series into (4.1), the first integral produces nonnegative powers and the second produces negative powers:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \quad (4.2)$$

The coefficients have one formula valid for every integer n :

$$a_n = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \quad r < \rho < R. \quad (4.3)$$

Contour deformation shows that the answer does not depend on the chosen radius within the annulus. The expansion is unique and converges on every compact subannulus. Its domain of convergence is an annulus, which may be a punctured disk when $r = 0$.

If the center is singular, the coefficients need not be derivatives of f there: those derivatives may not exist. In practice, algebraic rearrangements and known geometric or exponential series are often faster than evaluating the coefficient integrals individually.

4.2 Classification of isolated singularities

Definition 4.1. A point z_0 is an **isolated singularity** of f if f is holomorphic on some punctured disk $0 < |z - z_0| < R$ but does not yet have a holomorphic definition at z_0 .

The negative-power part of its Laurent series is called the **principal part**. It determines the classification.

1. If every negative-power coefficient vanishes, the singularity is **removable**. Define $f(z_0) = a_0$ to obtain a holomorphic extension. For example,

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

extends to the value one at zero.

2. If the principal part has finitely many nonzero terms,

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \dots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots, \quad a_{-m} \neq 0,$$

then z_0 is a **pole of order m** . A pole of order one is a simple pole. Equivalently, $(z - z_0)^m f(z)$ extends holomorphically to a nonzero value at z_0 .

3. If infinitely many negative-power coefficients are nonzero, z_0 is an **essential singularity**. The example

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

has an essential singularity at zero.

At a pole, $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$. An essential singularity does not have this uniform behavior: for $e^{1/z}$ the limit is infinite along positive real z and zero along negative real z .

4.3 Residues and how to compute them

Definition 4.2. The **residue** of f at an isolated singularity z_0 is the coefficient of $(z - z_0)^{-1}$ in its Laurent series:

$$\operatorname{Res}_{z=z_0} f(z) = a_{-1} = \frac{1}{2\pi i} \oint f(z) dz. \quad (4.4)$$

The contour in the last expression is a small positively oriented circle containing no other singularity.

If z_0 is a pole of order m , set

$$g(z) = (z - z_0)^m f(z).$$

The function g is holomorphic at z_0 , with Taylor expansion

$$g(z) = \sum_{j=0}^{\infty} \frac{g^{(j)}(z_0)}{j!} (z - z_0)^j.$$

After division by $(z - z_0)^m$, the power -1 occurs when $j = m - 1$. Therefore

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \Big|_{z=z_0}. \quad (4.5)$$

For a simple pole this reduces to

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

If $f = P/Q$ and Q has a simple zero at z_0 , then

$$Q(z) = Q'(z_0)(z - z_0) + O((z - z_0)^2),$$

so

$$\operatorname{Res}_{z=z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q'(z_0)}. \quad (4.6)$$

At an essential singularity the finite-order derivative formula is not available, but the coefficient definition still works. The residue of $e^{1/z}$ at zero is one.

4.4 The residue theorem

Theorem 4.3 (Residue theorem). *Suppose f is holomorphic on and inside a positively oriented simple closed curve Γ , except at finitely many isolated singularities $z_1, \dots, z_N$ inside the curve. Then*

$$\oint_{\Gamma} f(z) dz = 2\pi i \sum_{j=1}^N \operatorname{Res}_{z=z_j} f(z). \quad (4.7)$$

Proof. Remove small disjoint disks about the singularities. Cauchy's theorem on the remaining region gives the outer integral as the sum of counterclockwise integrals on the small circles. On a circle about z_j , integrate its Laurent series term by term. With $z - z_j = \rho e^{i\theta}$,

$$\oint (z - z_j)^n dz = i\rho^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta = \begin{cases} 2\pi i, & n = -1, \\ 0, & n \neq -1. \end{cases}$$

Every term vanishes except the residue term. Summing the small-circle integrals proves (4.7). $\square$

For clockwise contours the sign is reversed. For curves with multiple windings, each residue is multiplied by the corresponding winding number. The singularities must not lie on the integration path; real-axis poles require a limiting prescription.

5 Real integrals from semicircular contours

5.1 A rational integral worked from start to finish

Consider

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Use the complex function $f(z) = 1/(1+z^2)$ and close the real interval $[-R, R]$ by the upper semicircle C_R . The resulting contour is counterclockwise. Of the two poles $\pm i$, only i lies inside.

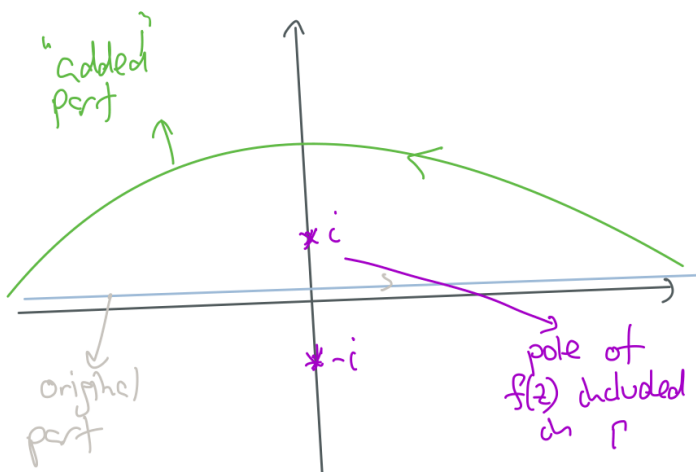


Figure 4: A real-axis integral closed by an upper semicircle.

The residue is

$$\text{Res}_{z=i} \frac{1}{1+z^2} = \lim_{z \rightarrow i} \frac{z-i}{(z-i)(z+i)} = \frac{1}{2i}.$$

Thus

$$\int_{-R}^R \frac{dx}{1+x^2} + \int_{C_R} \frac{dz}{1+z^2} = \pi.$$

The added arc must be checked rather than discarded without an estimate. On $|z| = R > 1$,

$$|1+z^2| \geq R^2 - 1, \quad \left| \int_{C_R} \frac{dz}{1+z^2} \right| \leq \frac{\pi R}{R^2 - 1} \rightarrow 0.$$

Taking $R \rightarrow \infty$ gives

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi. \quad (5.1)$$

For a rational function $P(z)/Q(z)$ with no real-axis pole, the same argument applies if

$$\deg Q \geq \deg P + 2.$$

The integrand is then $O(R^{-2})$ or smaller, while the semicircle has length πR , so its contribution tends to zero. The difficult step in a new integral is often choosing a contour whose added pieces can be controlled; the residue calculation itself is usually short.

5.2 Jordan's lemma and oscillatory integrals

For an integral containing e^{ikx} , the complex exponential provides decay in a suitable half-plane:

$$|e^{ikz}| = e^{-k \operatorname{Im} z}, \quad k \in \mathbb{R}.$$

For $k > 0$ use the upper half-plane; for $k < 0$ use the lower half-plane. This permits rational prefactors that decay only as $1/z$.

Lemma 5.1 (Jordan's lemma). *Let $k > 0$ and let f be continuous on an upper semicircle C_R . If*

$$M_R = \max_{0 \leq \theta \leq \pi} |f(Re^{i\theta})| \rightarrow 0,$$

then

$$\int_{C_R} f(z) e^{ikz} dz \rightarrow 0.$$

Proof. The parametrization $z = Re^{i\theta}$ gives

$$\begin{aligned} \left| \int_{C_R} f(z) e^{ikz} dz \right| &\leq M_R R \int_0^\pi e^{-kR \sin \theta} d\theta \\ &= 2M_R R \int_0^{\pi/2} e^{-kR \sin \theta} d\theta. \end{aligned}$$

On $0 \leq \theta \leq \pi/2$, concavity of sine gives $\sin \theta \geq 2\theta/\pi$. Therefore

$$\begin{aligned} \left| \int_{C_R} f(z) e^{ikz} dz \right| &\leq 2M_R R \int_0^{\pi/2} e^{-2kR\theta/\pi} d\theta \\ &= \frac{\pi M_R}{k} (1 - e^{-kR}) \leq \frac{\pi M_R}{k} \rightarrow 0. \end{aligned}$$

□

The estimate shows why it is important that M_R tends to zero; a fixed bound alone would not establish a vanishing arc. The lower half-plane version follows in the same way for $k < 0$.

If the real integral exists and the hypotheses hold, the residue formulas become

$$\int_{-\infty}^{\infty} f(x) e^{ikx} dx = \begin{cases} 2\pi i \sum_{\substack{\text{Im } z_j > 0}} \text{Res}_{z=z_j} [f(z) e^{ikz}], & k > 0, \\ -2\pi i \sum_{\substack{\text{Im } z_j < 0}} \text{Res}_{z=z_j} [f(z) e^{ikz}], & k < 0. \end{cases} \quad (5.2)$$

The minus sign in the second line comes from clockwise orientation.

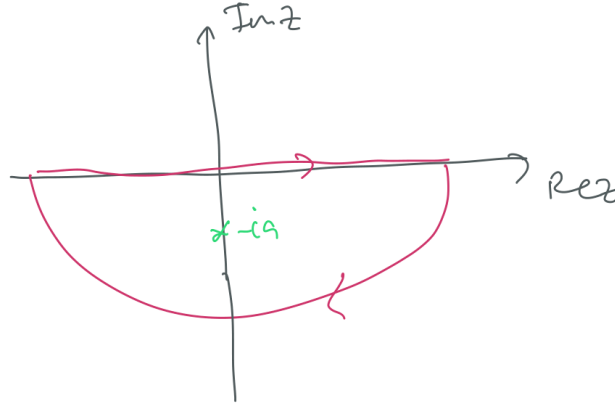


Figure 5: A lower contour has clockwise orientation when the real axis is traversed from left to right.

5.3 The cosine transform of a Lorentzian

For $a > 0$, define

$$J(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + a^2} dx.$$

When $k > 0$, the upper contour encloses only $z = ia$, and

$$\text{Res}_{z=ia} \frac{e^{ikz}}{z^2 + a^2} = \frac{e^{ik(ia)}}{2ia} = \frac{e^{-ka}}{2ia}.$$

Thus $J(k) = \pi e^{-ka}/a$. When $k < 0$, the lower contour encloses $z = -ia$. Its residue is

$$\frac{e^{ik(-ia)}}{-2ia} = -\frac{e^{ka}}{2ia},$$

and clockwise orientation gives $J(k) = \pi e^{ka}/a$. At $k = 0$, the rational contour estimate gives $J(0) = \pi/a$. All cases combine as

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + a^2} dx = \frac{\pi}{a} e^{-a|k|}. \quad (5.3)$$

The sine part is odd in x and integrates to zero. Taking the real part gives

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2 + a^2} dx = \frac{\pi}{a} e^{-a|k|}. \quad (5.4)$$

5.4 An oscillatory numerator and conditional convergence

Now consider

$$K(k) = \int_{-\infty}^{\infty} \frac{x \sin(kx)}{x^2 + a^2} dx.$$

For $k \neq 0$, the tails converge by oscillation, although the absolute integral generally diverges. The useful complex integrand is

$$\frac{ze^{ikz}}{z^2 + a^2}.$$

Its rational prefactor is $O(1/z)$, so the elementary length estimate is insufficient but Jordan's lemma applies.

For $k > 0$, the residue at ia is

$$\text{Res}_{z=ia} \frac{ze^{ikz}}{z^2 + a^2} = \frac{iae^{-ka}}{2ia} = \frac{1}{2} e^{-ka}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{xe^{ikx}}{x^2 + a^2} dx = \pi i e^{-ka}.$$

For $k < 0$, the residue at $-ia$ is $\frac{1}{2} e^{ka}$, and clockwise orientation gives $-\pi i e^{ka}$. Taking imaginary parts yields

$$K(k) = \pi \operatorname{sgn}(k) e^{-a|k|}, \quad k \neq 0. \quad (5.5)$$

At $k = 0$ the integrand is identically zero, so $K(0) = 0$. The one-sided limits as $k \rightarrow 0$ are not zero; convergence is not uniform there.

A second route differentiates (5.4) for $k \neq 0$:

$$\frac{d}{dk} \int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2 + a^2} dx = - \int_{-\infty}^{\infty} \frac{x \sin(kx)}{x^2 + a^2} dx.$$

One may justify this by first inserting an exponential convergence factor, differentiating the regularized integral, and then removing the factor away from $k = 0$. Differentiating $(\pi/a) e^{-a|k|}$ reproduces (5.5).

6 Unit-circle, rectangular, and sector contours

6.1 Converting trigonometric integrals to residues

For a full-period integral of a rational function of sine and cosine, set

$$z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Then z traverses the unit circle counterclockwise and

$$\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right), \quad \sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right), \quad d\theta = \frac{dz}{iz}. \quad (6.1)$$

Consequently,

$$\begin{aligned} & \int_0^{2\pi} F(\sin \theta, \cos \theta) d\theta \\ &= \frac{1}{i} \oint_{|z|=1} \frac{1}{z} F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) dz. \end{aligned} \quad (6.2)$$

If the transformed integrand has no pole on the unit circle, the residue theorem converts the integral into 2π times the sum of the residues of the expression following $1/i$.

Example 6.1 (The integral of sine squared). Consider $I = \int_0^{2\pi} \sin^2 \theta d\theta$. The rational function whose residues are required is

$$H(z) = \frac{1}{z} \left(\frac{z - z^{-1}}{2i} \right)^2 = -\frac{(z^2 - 1)^2}{4z^3}.$$

Its only pole inside the unit circle is at zero, of order three. Set

$$g(z) = z^3 H(z) = -\frac{1}{4}(z^2 - 1)^2.$$

Then

$$g'(z) = -z(z^2 - 1), \quad g''(z) = 1 - 3z^2,$$

so

$$\operatorname{Res}_{z=0} H(z) = \frac{g''(0)}{2!} = \frac{1}{2}.$$

Equation (6.2) therefore gives

$$I = \frac{1}{i} (2\pi i) \frac{1}{2} = \pi.$$

The direct identity $\sin^2 \theta = (1 - \cos 2\theta)/2$ gives the same answer. This simple example checks every factor of i in the contour substitution before applying it to harder rational integrals.

6.2 A rectangular contour adapted to an exponential

Consider

$$I(a) = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, \quad 0 < a < 1. \quad (6.3)$$

The conditions have a direct convergence meaning: as $x \rightarrow -\infty$ the integrand behaves as e^{ax} , and as $x \rightarrow +\infty$ it behaves as $e^{(a-1)x}$.

The denominator has imaginary period $2\pi i$. Choose

$$F(z) = \frac{e^{az}}{1+e^z}$$

and the counterclockwise rectangle with vertices $-R, R, R+2\pi i, -R+2\pi i$.

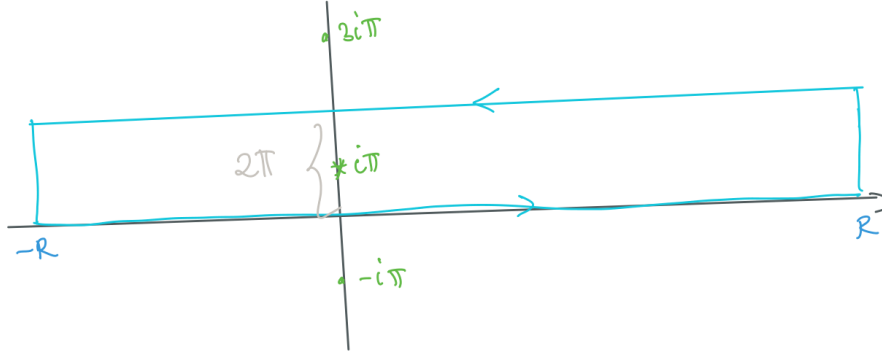


Figure 6: The rectangle exploits the imaginary period of the denominator.

On the top edge, $z = x + 2\pi i$ and x runs from R to $-R$. Since

$$F(x + 2\pi i) = e^{2\pi ia} F(x),$$

the sum of the bottom and top contributions is

$$(1 - e^{2\pi ia}) \int_{-R}^R \frac{e^{ax}}{1+e^x} dx.$$

The vertical contributions vanish. On the right edge $z = R + iy$,

$$|F(z)| \leq \frac{e^{aR}}{e^R - 1} = O(e^{-(1-a)R}),$$

and on the left edge $z = -R + iy$,

$$|F(z)| \leq \frac{e^{-aR}}{1 - e^{-R}} = O(e^{-aR}).$$

Both edges have length 2π , so these uniform bounds suffice.

The poles satisfy $e^z = -1$, or $z = (2n + 1)\pi i$. The only pole inside the strip is $z = \pi i$, and it is simple. By the quotient formula,

$$\text{Res}_{z=\pi i} F(z) = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}.$$

Thus

$$(1 - e^{2\pi i a}) I(a) = -2\pi i e^{\pi i a}.$$

Using

$$1 - e^{2\pi i a} = -e^{\pi i a} (e^{\pi i a} - e^{-\pi i a}) = -2i e^{\pi i a} \sin(\pi a),$$

we obtain

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} dx = \frac{\pi}{\sin(\pi a)}, \quad 0 < a < 1. \quad (6.4)$$

6.3 Why sector contours are useful

A sector contour consists of two rays from the origin and a joining arc. It is useful when rotating z through a fixed angle changes the integrand by a simple factor. The integral along the second ray can then be expressed in terms of the first.

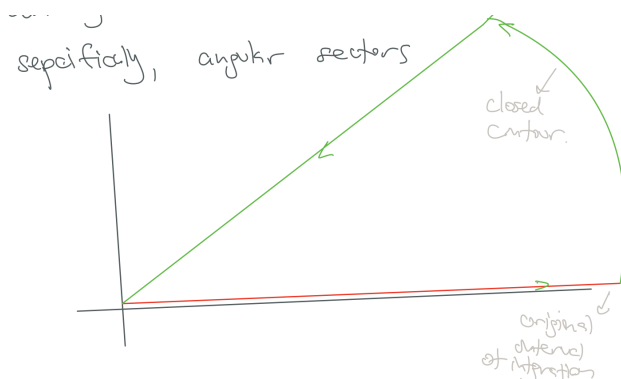


Figure 7: A sector contour relates integrals along two rays.

For example, suppose $F(e^{i\alpha}x)$ is easily related to $F(x)$. On the outward ray $z = x$, $0 \leq x \leq R$, the integral is $\int_0^R F(x) dx$. On the returning ray $z = e^{i\alpha}x$, $R \geq x \geq 0$, it is

$$-e^{i\alpha} \int_0^R F(e^{i\alpha}x) dx.$$

The factor $e^{i\alpha}$ comes from dz and must be retained. After showing that the large arc vanishes, the two ray integrals are related to the residues in the sector.

Exercise 6.2 (Using a sector). Apply a sector of opening angle $2\pi/n$ to $F(z) = 1/(1 + z^n)$, with integer $n > 1$, to evaluate $\int_0^\infty dx/(1 + x^n)$.

Solution. The rotated denominator is unchanged because $(e^{2\pi i/n}x)^n = x^n$. The large arc contribution is $O(R^{1-n})$ and vanishes. The contour encloses the simple pole $z_0 = e^{\pi i/n}$, whose residue is

$$\frac{1}{nz_0^{n-1}} = -\frac{z_0}{n}.$$

Writing the desired integral as I ,

$$(1 - e^{2\pi i/n})I = -\frac{2\pi i}{n}e^{\pi i/n}.$$

Since $1 - e^{2\pi i/n} = -2ie^{\pi i/n} \sin(\pi/n)$,

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n} \csc \frac{\pi}{n}.$$

□

7 Branches, Riemann surfaces, and keyhole contours

7.1 Why the logarithm needs a branch

For $z = re^{i\theta} \neq 0$, every number

$$w_n = \log r + i(\theta + 2\pi n), \quad n \in \mathbb{Z},$$

satisfies $e^{w_n} = z$. Thus the inverse of the exponential is multivalued on the punctured plane. Going once around the origin changes a continuously followed argument by 2π , and changes the logarithm by $2\pi i$.

To obtain a single-valued holomorphic logarithm, remove a curve from the branch point to infinity and select an interval of length 2π for the argument. Two common choices are

$$-\pi < \arg z < \pi \quad (\text{cut along the negative real axis}),$$

and

$$0 < \arg z < 2\pi \quad (\text{cut along the positive real axis}).$$

On either cut domain,

$$\text{Log } z = \log |z| + i \arg z, \quad \frac{d}{dz} \text{Log } z = \frac{1}{z}.$$

The cut is a choice of domain for a branch. It is not a line of intrinsic singular points: away from the branch point, each point has a neighborhood on which a logarithm can be defined. Different branches differ by an integer multiple of $2\pi i$ on overlapping connected regions.

7.2 Fractional powers and sheets

Define a chosen branch of a complex power by

$$z^\alpha = \exp(\alpha \operatorname{Log} z). \quad (7.1)$$

A turn around the origin multiplies the continued value by $e^{2\pi i \alpha}$. For $\alpha = 1/2$ this is -1 , so a square root changes sign after one turn and returns to its original value after two. For a rational $\alpha = p/q$ in lowest terms, there are q distinct values. An irrational real exponent gives infinitely many distinct values under repeated turns.

Instead of cutting a single plane, one may keep track of the turns using a **Riemann surface**. For the logarithm, infinitely many sheets are joined so that crossing a cut moves to the neighboring sheet. A point on the surface records both the nonzero complex number and the chosen continuously followed argument. The logarithm is single-valued on this surface. Points on different sheets that project to the same z have logarithms differing by $2\pi i n$. The square-root surface has two sheets joined at the branch point; the square root becomes single-valued when the sheet is included in the specification of the point.

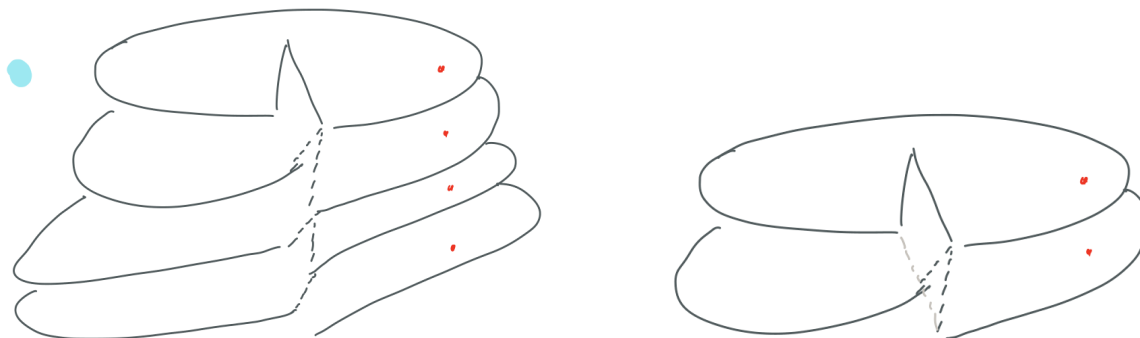


Figure 8: Original sketches of multiple sheets for the logarithm (left) and the square root (right).

7.3 A keyhole integral with every contour contribution

Consider

$$I(\alpha) = \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx, \quad 0 < \operatorname{Re} \alpha < 1. \quad (7.2)$$

For positive x , define $x^{\alpha-1} = e^{(\alpha-1) \log x}$ using the ordinary real logarithm. Near zero its magnitude behaves as $x^{\operatorname{Re} \alpha - 1}$, which is integrable if $\operatorname{Re} \alpha > 0$. At infinity it behaves as $x^{\operatorname{Re} \alpha - 2}$, which is integrable if $\operatorname{Re} \alpha < 1$.

Use

$$F(z) = \frac{z^{\alpha-1}}{1+z}, \quad 0 < \arg z < 2\pi.$$

The branch cut lies along the positive real axis. Choose the keyhole contour which runs from ε to R just above the cut, travels counterclockwise around the large circle,

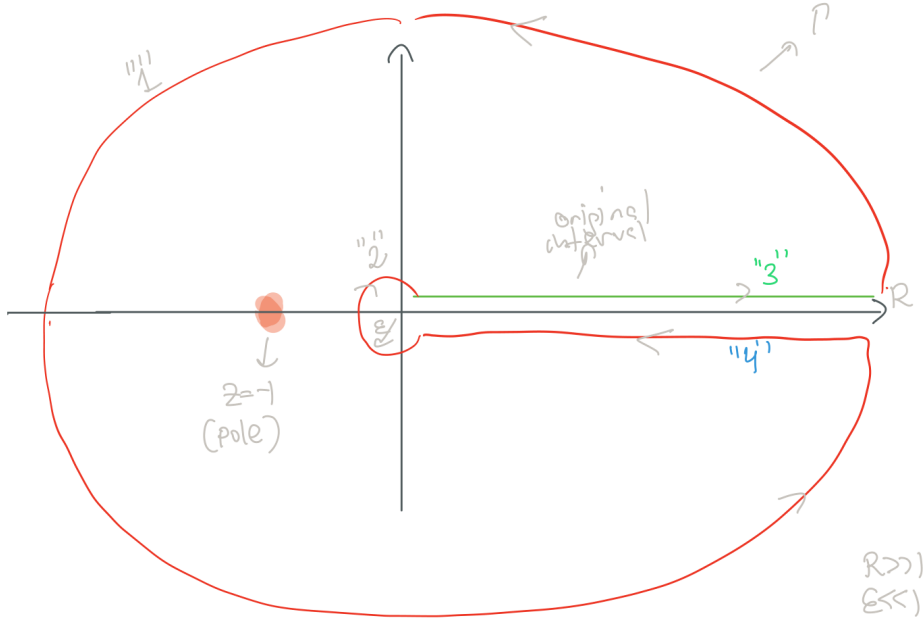


Figure 9: The two sides of a branch cut carry different values of the power.

returns from R to ε just below the cut, and finally travels clockwise around the small circle.

On the upper side, $\arg z \rightarrow 0$, so the contribution is

$$I_{\varepsilon,R} = \int_{\varepsilon}^R \frac{x^{\alpha-1}}{1+x} dx.$$

On the lower side, $\arg z \rightarrow 2\pi$, so

$$z^{\alpha-1} = x^{\alpha-1} e^{2\pi i(\alpha-1)}.$$

The orientation is reversed there, and its contribution is

$$-e^{2\pi i(\alpha-1)} I_{\varepsilon,R}.$$

For fixed α , the factor $|e^{i(\alpha-1)\theta}|$ is bounded for $0 \leq \theta \leq 2\pi$. Thus, with a constant C_α independent of R, ε ,

$$\begin{aligned} \left| \int_{|z|=R} F(z) dz \right| &\leq C_\alpha R^{\operatorname{Re} \alpha - 2} (2\pi R) = O(R^{\operatorname{Re} \alpha - 1}) \rightarrow 0, \\ \left| \int_{|z|=\varepsilon} F(z) dz \right| &\leq C_\alpha \varepsilon^{\operatorname{Re} \alpha - 1} (2\pi \varepsilon) = O(\varepsilon^{\operatorname{Re} \alpha}) \rightarrow 0. \end{aligned}$$

The same inequalities that ensure convergence of the original real integral ensure that the two circular pieces vanish.

The only enclosed pole is at $z = -1$. On the chosen branch its argument is π , hence

$$\operatorname{Res}_{z=-1} F(z) = (-1)^{\alpha-1} = e^{\pi i(\alpha-1)}.$$

The residue theorem gives

$$[1 - e^{2\pi i(\alpha-1)}] I(\alpha) = 2\pi i e^{\pi i(\alpha-1)}.$$

Use

$$\begin{aligned} 1 - e^{2\pi i(\alpha-1)} &= -2ie^{\pi i(\alpha-1)} \sin(\pi(\alpha-1)) \\ &= 2ie^{\pi i(\alpha-1)} \sin(\pi\alpha) \end{aligned}$$

to obtain

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin(\pi\alpha)}, \quad 0 < \operatorname{Re} \alpha < 1. \quad (7.3)$$

For real $\alpha = a$, the substitution $x = e^t$ transforms this into (6.3), explaining why the rectangular and keyhole calculations yield the same expression.

8 Principal values and real-axis singularities

8.1 An improper integral is not the same as a principal value

A singularity inside an interval requires separate one-sided limits for the ordinary improper integral. For example,

$$\int_{-b}^a \frac{dx}{x}, \quad a, b > 0,$$

does not exist as an ordinary improper integral: the two one-sided integrals diverge. One must not add $+\infty$ and $-\infty$.

A different object results if the two sides approach the singularity symmetrically. Its **Cauchy principal value** is

$$\text{PV} \int_{-b}^a \frac{f(x)}{x} dx = \lim_{\varepsilon \downarrow 0} \left[\int_{-b}^{-\varepsilon} \frac{f(x)}{x} dx + \int_{\varepsilon}^a \frac{f(x)}{x} dx \right], \quad (8.1)$$

provided the symmetric limit exists. For $f = 1$,

$$\int_{-b}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^a \frac{dx}{x} = \log \varepsilon - \log b + \log a - \log \varepsilon = \log \frac{a}{b}.$$

Therefore the principal value is finite even though the ordinary improper integral diverges.

If f is continuously differentiable near zero, write

$$\frac{f(x)}{x} = \frac{f(0)}{x} + \frac{f(x) - f(0)}{x}.$$

The second term extends continuously to $f'(0)$. Hence

$$\text{PV} \int_{-b}^a \frac{f(x)}{x} dx = f(0) \log \frac{a}{b} + \int_{-b}^a \frac{f(x) - f(0)}{x} dx. \quad (8.2)$$

This displays directly why a simple pole has a natural symmetric regularization. For a pole at x_0 , replace x by $x - x_0$. At infinite endpoints, sufficient decay or a separately specified principal-value limit at infinity is also required.

8.2 Indenting a contour above or below a simple pole

Let f be holomorphic near zero and meromorphic in the upper half-plane, with no other real-axis singularity. Assume the large arc contribution of $f(z)/z$ vanishes. To avoid the pole at zero, replace $[-\varepsilon, \varepsilon]$ by an upper semicircle traversed from left to right:

$$z = \varepsilon e^{i\theta}, \quad \pi \geq \theta \geq 0.$$

Its contribution is

$$\int_{\text{upper indentation}} \frac{f(z)}{z} dz = i \int_{\pi}^0 f(\varepsilon e^{i\theta}) d\theta \longrightarrow -\pi i f(0).$$

The small indentation is clockwise and excludes zero from the interior. The residue theorem therefore gives

$$\text{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x} dx - \pi i f(0) = 2\pi i \sum_{\text{Im } z_j > 0} \text{Res}_{z=z_j} \frac{f(z)}{z}. \quad (8.3)$$

Equivalently,

$$\text{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x} dx = 2\pi i \sum_{\text{Im } z_j > 0} \text{Res}_{z=z_j} \frac{f(z)}{z} + \pi i f(0). \quad (8.4)$$

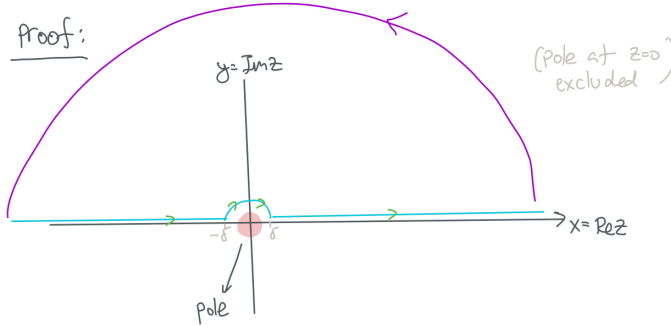


Figure 10: An upper indentation excludes the real-axis pole.

For a lower indentation from $-\varepsilon$ to ε , take θ from $-\pi$ to zero. Its contribution tends to $+\pi i f(0)$. The contour now encloses zero, so the residue sum contains the extra residue $f(0)$:

$$\text{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x} dx + \pi i f(0) = 2\pi i \left[\sum_{\text{Im } z_j > 0} \text{Res}_{z=z_j} \frac{f(z)}{z} + f(0) \right].$$

Rearranging reproduces (8.4). The principal value is independent of this choice; the indentation contribution and the set of enclosed poles must be changed together.

The half-circle value is determined by its side and orientation. Changing to a path with extra winding about the pole adds integer multiples of $2\pi i f(0)$; an arbitrary path need not have the same value.

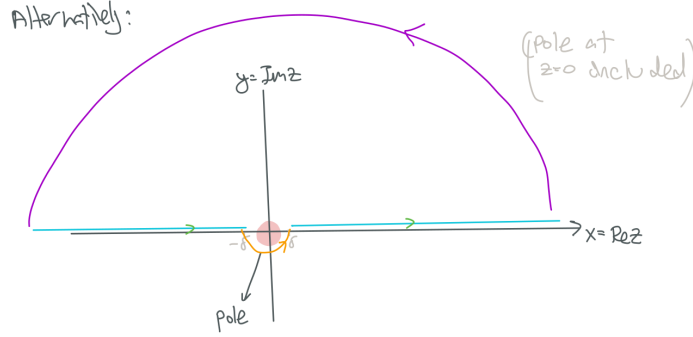


Figure 11: A lower indentation includes the real-axis pole inside the contour.

8.3 Why higher-order poles require care

The generic integral $f(x)/x^n$ for $n > 1$ does not acquire a finite Cauchy principal value just by deleting equal intervals. For instance,

$$\int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \int_{\varepsilon}^1 \frac{dx}{x^2} = \frac{2}{\varepsilon} - 2 \longrightarrow \infty.$$

There can be cancellations for special numerators and parity. The principal value of $1/x^3$ on $[-1, 1]$ is zero, but that of $(1+x)/x^3$ diverges because its even $1/x^2$ term survives. More generally, the Taylor expansion of f produces several singular powers whose parity determines which divergent terms cancel.

The indentation calculation makes the same distinction. For a monomial,

$$\int_{\text{upper indentation}} \frac{dz}{z^n} = i\varepsilon^{1-n} \int_{\pi}^0 e^{i(1-n)\theta} d\theta.$$

For $n = 1$ the result is the finite value $-\pi i$. For higher powers it can contain divergent terms; special powers can have zero angular integral, but this does not remove the generic divergences arising from a nonconstant numerator. Subtracting these divergent terms defines a *Hadamard finite part*, a different regularization from the ordinary Cauchy principal value.

8.4 A smooth regularization of the principal value

For a sufficiently smooth, decaying test function φ , one has

$$\text{PV} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x - x_0} dx = \lim_{\eta \downarrow 0} \int_{-\infty}^{\infty} \frac{(x - x_0)\varphi(x)}{(x - x_0)^2 + \eta^2} dx. \quad (8.5)$$

To see the local reason, set $y = x - x_0$ and subtract $\varphi(x_0)$ on a small symmetric interval. The constant term integrates to zero because $y/(y^2 + \eta^2)$ is odd. The remaining numerator is $\varphi(x_0 + y) - \varphi(x_0) = O(y)$, so its product with $y/(y^2 + \eta^2)$ is bounded near zero and tends to $[\varphi(x_0 + y) - \varphi(x_0)]/y$. Away from zero, ordinary dominated convergence applies. This reproduces the subtraction formula (8.2).

8.5 The Sokhotski–Plemelj identities

For $\eta > 0$, elementary algebra gives

$$\frac{1}{x - x_0 \pm i\eta} = \frac{x - x_0}{(x - x_0)^2 + \eta^2} \mp i \frac{\eta}{(x - x_0)^2 + \eta^2}. \quad (8.6)$$

The first term tends to the principal-value distribution by (8.5). The second term is concentrated near $x = x_0$. Indeed, with $x - x_0 = \eta t$,

$$\int_{-\infty}^{\infty} \frac{\eta \varphi(x)}{(x - x_0)^2 + \eta^2} dx = \int_{-\infty}^{\infty} \frac{\varphi(x_0 + \eta t)}{1 + t^2} dt \longrightarrow \pi \varphi(x_0).$$

Thus

$$\frac{1}{\pi} \frac{\eta}{(x - x_0)^2 + \eta^2} \longrightarrow \delta(x - x_0)$$

in the sense of distributions. Here the delta distribution is defined by $\int \delta(x - x_0) \varphi(x) dx = \varphi(x_0)$; it is not an ordinary function with a finite value at x_0 .

Combining the two limits gives the **Sokhotski–Plemelj identities**:

$$\frac{1}{x - x_0 \pm i0} = \text{PV} \frac{1}{x - x_0} \mp i\pi \delta(x - x_0). \quad (8.7)$$

Their precise meaning is the integrated statement

$$\lim_{\eta \downarrow 0} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x - x_0 \pm i\eta} dx = \text{PV} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x - x_0} dx \mp i\pi \varphi(x_0). \quad (8.8)$$

The infinitesimal sign records from which side the pole approaches the real axis. It carries information and cannot be dropped before performing the integral.

8.6 Boundary values of analytic functions

If f is holomorphic throughout the upper half-plane, continuous on the real axis, and decays sufficiently for the large contour to vanish, there are no off-axis residues in (8.4). Translating the real pole to x_0 gives

$$\text{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx = \pi i f(x_0), \quad f(x_0) = \frac{1}{\pi i} \text{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx. \quad (8.9)$$

This is the boundary form of Cauchy's formula. Separating its real and imaginary parts produces Hilbert-transform relations. Applied to the Fourier transform of a causal response, these become the Kramers–Kronig dispersion relations studied in the linear-response part of the course.

Part II

Vector and Function Spaces

Function
spaces

9 Inner products, orthogonality, and approximation

9.1 Why functions should be regarded as vectors

A differential equation can be viewed as an equation involving a linear operator. For example,

$$\mathcal{L}f = p_0(x)f^{(n)} + p_1(x)f^{(n-1)} + \cdots + p_n(x)f$$

resembles multiplication of a vector by a matrix. Its eigenvalue problem is

$$\mathcal{L}u = \lambda u.$$

The main questions are which values of λ are allowed, what the corresponding functions u look like, and whether these eigenfunctions can represent an arbitrary function of interest. The answers provide the basis for separation of variables, Green's functions, and quantum mechanics. We first develop the vector-space language needed to state these questions.

A vector space is closed under addition and scalar multiplication. In $\mathbb{C}^N$, the usual inner product is

$$\langle u, v \rangle = \sum_{j=1}^N u_j^* v_j.$$

Throughout these notes, an inner product is conjugate linear in its first argument and linear in its second. Thus

$$\begin{aligned}\langle \alpha u + \beta v, z \rangle &= \alpha^* \langle u, z \rangle + \beta^* \langle v, z \rangle, \\ \langle u, \alpha v + \beta z \rangle &= \alpha \langle u, v \rangle + \beta \langle u, z \rangle, \\ \langle u, v \rangle &= \langle v, u \rangle^*, \quad \langle u, u \rangle \geq 0.\end{aligned}$$

Positive definiteness means $\langle u, u \rangle = 0$ only for the zero vector. The norm and distance are

$$\|u\| = \sqrt{\langle u, u \rangle}, \quad d(u, v) = \|u - v\|.$$

In particular,

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2 \operatorname{Re} \langle u, v \rangle.$$

For real vectors, this is the familiar Euclidean distance formula.

9.2 Cauchy–Schwarz, the triangle inequality, and continuity

Theorem 9.1 (Cauchy–Schwarz inequality). *For any two vectors in an inner-product space,*

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

Equality holds exactly when the vectors are linearly dependent, including cases in which one vector is zero.

Proof. If $u = 0$, the statement is immediate. Otherwise subtract from v its component along u :

$$r = v - u \frac{\langle u, v \rangle}{\|u\|^2}.$$

Expansion of its squared norm gives

$$0 \leq \|r\|^2 = \|v\|^2 - \frac{|\langle u, v \rangle|^2}{\|u\|^2}.$$

Multiplying by $\|u\|^2$ gives the inequality. Equality is equivalent to $r = 0$, which is precisely the condition that v is proportional to u . $\square$

The triangle inequality follows by expanding the squared norm and using Cauchy–Schwarz:

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + 2 \operatorname{Re} \langle u, v \rangle + \|v\|^2 \\ &\leq \|u\|^2 + 2 \|u\| \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2. \end{aligned}$$

Taking square roots yields $\|u + v\| \leq \|u\| + \|v\|$. A useful consequence is the reverse triangle inequality,

$$|\|u\| - \|v\|| \leq \|u - v\|.$$

Indeed, $\|u\| \leq \|u - v\| + \|v\|$ gives one direction, and interchanging u and v gives the other.

The inner product is continuous in the norm topology. If $u_n \rightarrow u$, then for every fixed v ,

$$|\langle u_n, v \rangle - \langle u, v \rangle| = |\langle u_n - u, v \rangle| \leq \|u_n - u\| \|v\| \rightarrow 0.$$

This simple estimate will justify extracting expansion coefficients from limits of functions.

9.3 Completeness of a normed space

Definition 9.2. A sequence u_n is **Cauchy** if, for each $\epsilon > 0$, there is an N such that

$$\|u_n - u_m\| < \epsilon \quad (m, n > N).$$

A normed space is **complete** if every Cauchy sequence converges in norm to an element of that space. A complete inner-product space is a **Hilbert space**.

Convergence of successive differences to zero is insufficient: the partial sums of the harmonic series have successive differences $1/n$ but do not form a Cauchy sequence. Completeness concerns all sufficiently late pairs of terms.

Finite-dimensional real and complex inner-product spaces are complete. The rational numbers, with the ordinary distance, give a simple example of an incomplete metric space: rational approximations to $\sqrt{2}$ form a Cauchy sequence, but their limit is not rational. In function spaces, we must similarly decide whether limits remain inside the chosen space.

9.4 Orthonormal bases and Gram–Schmidt

A basis $v_1, \dots, v_N$ is a linearly independent spanning set. Every vector has a unique expansion in any basis. Orthogonality makes the coefficients especially easy to find. If

$$\langle e_i, e_j \rangle = \delta_{ij}, \quad v = \sum_{j=1}^N c_j e_j,$$

then taking the inner product with e_i gives

$$c_i = \langle e_i, v \rangle.$$

For an orthogonal but unnormalized basis, the corresponding formula is $c_i = \langle v_i, v \rangle / \|v_i\|^2$.

Starting with linearly independent $v_1, v_2, \dots$, Gram–Schmidt constructs an orthonormal set with the same successive spans:

$$\begin{aligned} r_1 &= v_1, & e_1 &= \frac{r_1}{\|r_1\|}, \\ r_2 &= v_2 - e_1 \langle e_1, v_2 \rangle, & e_2 &= \frac{r_2}{\|r_2\|}, \\ r_n &= v_n - \sum_{j=1}^{n-1} e_j \langle e_j, v_n \rangle, & e_n &= \frac{r_n}{\|r_n\|}. \end{aligned}$$

For $i < n$,

$$\langle e_i, r_n \rangle = \langle e_i, v_n \rangle - \sum_{j < n} \delta_{ij} \langle e_j, v_n \rangle = 0.$$

Normalization then gives $\langle e_n, e_n \rangle = 1$. Linear independence ensures $r_n \neq 0$; otherwise v_n would already belong to the span of the earlier vectors. At each step,

$$\text{span}\{e_1, \dots, e_n\} = \text{span}\{v_1, \dots, v_n\}.$$

Thus orthogonalization changes the coordinates used to describe a subspace, but not the subspace itself.

9.5 The best-approximation theorem

Let $e_1, \dots, e_k$ be orthonormal, and consider approximation of v by vectors in $V_k = \text{span}\{e_1, \dots, e_k\}$. Define its orthogonal projection

$$P_k v = \sum_{j=1}^k \langle e_j, v \rangle e_j.$$

The residual $r = v - P_k v$ satisfies $\langle e_j, r \rangle = 0$ for all $j \leq k$. For any trial approximation $w = \sum_{j=1}^k b_j e_j$,

$$v - w = (v - P_k v) + (P_k v - w).$$

The two terms on the right are orthogonal, so Pythagoras gives

$$\|v - w\|^2 = \|v - P_k v\|^2 + \sum_{j=1}^k |\langle e_j, v \rangle - b_j|^2. \quad (9.1)$$

The second term is nonnegative and vanishes only when $b_j = \langle e_j, v \rangle$ for every j . Therefore $P_k v$ is the unique best approximation in V_k .

If $v = \sum_{j=1}^N c_j e_j$, the minimum squared error is

$$\|v - P_k v\|^2 = \sum_{j=k+1}^N |c_j|^2.$$

Increasing the number of orthonormal basis vectors leaves the previously computed coefficients unchanged. This property is useful when deciding how many terms to keep in a numerical approximation.

9.6 Parseval's identity in finite dimensions

Using orthonormality directly,

$$\|v\|^2 = \left\langle \sum_i c_i e_i, \sum_j c_j e_j \right\rangle = \sum_{i,j} c_i^* c_j \delta_{ij} = \sum_j |c_j|^2.$$

This is Parseval's identity. It says that the squared length is unchanged when a vector is represented by its coefficients in an orthonormal basis. The same argument gives

$$\langle u, v \rangle = \sum_j \langle e_j, u \rangle^* \langle e_j, v \rangle.$$

The infinite-dimensional versions require convergence, which is the next issue to address.

10 Hilbert spaces of functions

10.1 From a discretized interval to an integral

Sample a function at $x_1, \dots, x_N \in [a, b]$ and regard $(f(x_1), \dots, f(x_N))$ as a vector. To obtain a meaningful continuum limit, include the mesh size in the inner product:

$$\sum_{j=1}^N f(x_j)^* g(x_j) \Delta x \longrightarrow \int_a^b f(x)^* g(x) dx.$$

A positive weight allows different positions to contribute differently:

$$\langle f, g \rangle_w = \int_a^b f(x)^* g(x) w(x) dx, \quad \|f\|_w^2 = \int_a^b |f(x)|^2 w(x) dx. \quad (10.1)$$

We assume w is positive almost everywhere and sufficiently regular for the operations in question. The space $L_w^2(a, b)$ consists of functions with finite squared norm, identifying functions that agree almost everywhere. This identification is necessary: changing a function at one point does not change the integral and hence does not change the vector it represents.

For $w = 1$, we write $L^2(a, b)$. This space is complete. Cauchy–Schwarz, the triangle inequality, and the projection argument therefore carry over without changing their proofs. An element of L^2 need not be continuous, and an arbitrary L^2 element has no intrinsically specified value at an individual point.

10.2 Orthonormal expansions, Bessel’s inequality, and Parseval

Suppose $u_1, u_2, \dots$ are orthonormal in $L_w^2(a, b)$ and put

$$c_n = \langle u_n, f \rangle_w, \quad S_N f = \sum_{n=1}^N c_n u_n.$$

The same residual calculation as in finite dimensions yields

$$\|f - S_N f\|_w^2 = \|f\|_w^2 - \sum_{n=1}^N |c_n|^2 \geq 0. \quad (10.2)$$

Thus

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \|f\|_w^2,$$

which is **Bessel’s inequality**. The inequality alone does not say that the chosen functions span the entire space.

Definition 10.1. An orthonormal set is **complete** if its finite linear combinations are dense in the Hilbert space. Equivalently, the only vector orthogonal to every member of the set is the zero vector.

For a complete orthonormal set, $S_N f \rightarrow f$ in norm. To see why, choose a finite linear combination g with $\|f - g\|_w < \epsilon$. Once N includes all basis functions appearing in g , the best-approximation property gives

$$\|f - S_N f\|_w \leq \|f - g\|_w < \epsilon.$$

Taking $N \rightarrow \infty$ in (10.2) proves

$$f = \sum_{n=1}^{\infty} \langle u_n, f \rangle_w u_n \quad \text{in } L_w^2, \quad \|f\|_w^2 = \sum_{n=1}^{\infty} |\langle u_n, f \rangle_w|^2. \quad (10.3)$$

The error from retaining only N terms is exactly

$$\|f - S_N f\|_w^2 = \sum_{n>N} |c_n|^2.$$

Conversely, if equality holds in Parseval for every f , this error tends to zero and the orthonormal set is complete.

The word “basis” here refers to an orthonormal Hilbert-space basis: generally infinitely many terms are needed. The equality in (10.3) is an equality in norm, not an automatic assertion about convergence at every point.

10.3 Three useful Fourier bases

On $(0, 1)$, the normalized sine functions are

$$s_n(x) = \sqrt{2} \sin(n\pi x), \quad n = 1, 2, \dots$$

Their orthogonality follows from

$$2 \sin(n\pi x) \sin(m\pi x) = \cos((n - m)\pi x) - \cos((n + m)\pi x).$$

For $n \neq m$, both cosines integrate to zero on $(0, 1)$; for $n = m$, $2 \int_0^1 \sin^2(n\pi x) dx = 1$. The sine system is a complete basis of $L^2(0, 1)$.

Another complete orthonormal basis is

$$c_0(x) = 1, \quad c_n(x) = \sqrt{2} \cos(n\pi x), \quad n \geq 1.$$

The constant is needed for completeness. The sine system and the cosine system are separate bases; their union is not an orthogonal set on $(0, 1)$.

For periodic expansions, use

$$e_n(x) = e^{2\pi i n x}, \quad n \in \mathbb{Z}.$$

Indeed,

$$\langle e_m, e_n \rangle = \int_0^1 e^{2\pi i(n-m)x} dx = \delta_{mn}.$$

On a general interval of length L , the normalized exponentials become $L^{-1/2} e^{2\pi i n(x-a)/L}$. Boundary conditions in differential problems determine which of these representations is most convenient.

10.4 A complete worked sine-series example

Consider the representative

$$f(x) = 1 \quad (0 < x < 1), \quad f(0) = f(1) = 0.$$

Its sine coefficients are

$$\begin{aligned} c_n &= \int_0^1 \sqrt{2} \sin(n\pi x) dx \\ &= \frac{\sqrt{2}}{n\pi} (1 - (-1)^n) = \begin{cases} 0, & n \text{ even}, \\ \frac{2\sqrt{2}}{n\pi}, & n \text{ odd}. \end{cases} \end{aligned}$$

Hence

$$1 = \frac{4}{\pi} \left(\sin \pi x + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \cdots \right) \quad (0 < x < 1). \quad (10.4)$$

For N odd modes, write

$$S_N(x) = \frac{4}{\pi} \sum_{j=0}^{N-1} \frac{\sin((2j+1)\pi x)}{2j+1}.$$

Its exact squared error is

$$\|f - S_N\|^2 = 1 - \frac{8}{\pi^2} \sum_{j=0}^{N-1} \frac{1}{(2j+1)^2} \rightarrow 0.$$

In particular, Parseval gives the familiar sum $\sum_{j \geq 0} (2j+1)^{-2} = \pi^2/8$.

Every partial sum is smooth and vanishes at both endpoints. Yet for each fixed interior point the limit is one. The two limiting operations do not commute:

$$\lim_{x \rightarrow 0^+} \lim_{N \rightarrow \infty} S_N(x) = 1, \quad \lim_{N \rightarrow \infty} \lim_{x \rightarrow 0^+} S_N(x) = 0.$$

This rules out uniform convergence on the closed interval.

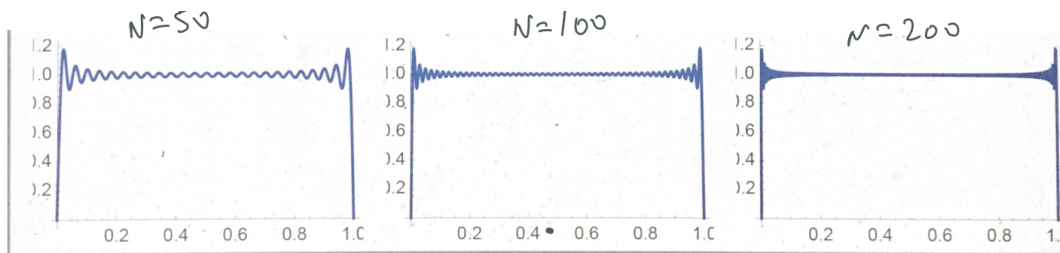


Figure 12: Sine partial sums approaching a constant in the interior. The oscillatory region narrows near the endpoints as more terms are retained.

The endpoint oscillations illustrate the **Gibbs phenomenon**. For a Fourier expansion of a function with a jump, the width of the oscillatory region decreases as more terms are added, but the limiting relative overshoot remains nonzero. Thus a sequence can converge in mean square while retaining large deviations in a shrinking region. For the sine series, the relevant periodic extension is odd and has jumps at the endpoints. Away from these jumps the approximation improves rapidly.

10.5 Why continuity does not follow from mean-square convergence

The endpoint example shows that the space of continuous functions satisfying $f(0) = f(1) = 0$ is not complete in the L^2 norm: its limit is the constant-one equivalence class, which has no continuous representative with those endpoint values.

To see the same issue for all continuous functions, approximate the interior step

$$h(x) = \begin{cases} 0, & 0 < x < \frac{1}{2}, \\ 1, & \frac{1}{2} < x < 1 \end{cases}$$

by continuous functions h_ϵ that equal zero below $\frac{1}{2} - \epsilon$, rise linearly from zero to one over a region of width 2ϵ , and equal one above $\frac{1}{2} + \epsilon$. The difference is supported in that region and has magnitude at most one, so

$$\|h_\epsilon - h\|_{L^2}^2 \leq 2\epsilon \longrightarrow 0.$$

The limit has no continuous representative. Hence $C[0, 1]$ is not complete when equipped with the L^2 norm, although it is complete with the uniform norm. Completeness depends on both the set of objects and the norm used to measure distance.

Point evaluation is not continuous with respect to the L^2 norm. For example, a triangular peak of height one and width 2ϵ centered at x_0 has norm tending to zero while its value at x_0 remains one. This distinction is essential when applying delta distributions and imposing boundary conditions on operators.

11 Orthogonal polynomial bases

11.1 Why a monomial fit changes when its degree changes

The monomials $1, x, x^2, \dots$ are linearly independent, but they are not orthogonal. On $[0, 1]$ with unit weight,

$$\langle x^m, x^n \rangle = \int_0^1 x^{m+n} dx = \frac{1}{m+n+1}.$$

More generally, a positive weight cannot make all the monomials orthogonal: for example, $\langle 1, x^2 \rangle_w = \int x^2 w(x) dx > 0$ on a nontrivial interval.

To minimize

$$E = \left\| f - \sum_{j=0}^N a_j x^j \right\|_w^2,$$

the residual must be orthogonal to every monomial in the trial space:

$$\left\langle x^m, f - \sum_{j=0}^N a_j x^j \right\rangle_w = 0, \quad 0 \leq m \leq N.$$

Therefore the coefficients solve the **normal equations**

$$\sum_{j=0}^N G_{mj} a_j = b_m, \quad G_{mj} = \langle x^m, x^j \rangle_w, \quad b_m = \langle x^m, f \rangle_w. \quad (11.1)$$

The Gram matrix G is not diagonal. Adding higher powers changes the system and generally changes coefficients of lower powers as well.

Example 11.1 (Approximating x^2 on the unit interval). The best constant is its mean, $a_0 = \int_0^1 x^2 dx = 1/3$. For a linear approximation $a + bx$, the normal equations are

$$a + \frac{b}{2} = \frac{1}{3}, \quad \frac{a}{2} + \frac{b}{3} = \frac{1}{4}.$$

Subtract half the first equation from the second: $b(1/3 - 1/4) = 1/4 - 1/6$, so $b = 1$ and $a = -1/6$. The constant coefficient changes from $1/3$ to $-1/6$. Allowing degree two gives the exact polynomial x^2 , whose constant and linear coefficients are both zero.

11.2 Gram–Schmidt for polynomials

Apply the construction to $1, x, x^2, \dots$ in $L_w^2(a, b)$:

$$q_0(x) = \frac{1}{\sqrt{\int_a^b w(x) dx}},$$

$$r_n(x) = x^n - \sum_{j=0}^{n-1} q_j(x) \langle q_j, x^n \rangle_w, \quad q_n = \frac{r_n}{\|r_n\|_w}.$$

Each q_n has degree exactly n , since lower-degree terms cannot cancel the leading term of x^n . The polynomials satisfy

$$\langle q_n, q_m \rangle_w = \delta_{nm}.$$

An equivalent recursive construction starts from xq_n :

$$q_{n+1} = \frac{xq_n - \sum_{j=0}^n q_j \langle q_j, xq_n \rangle_w}{\left\| xq_n - \sum_{j=0}^n q_j \langle q_j, xq_n \rangle_w \right\|_w}.$$

The numerator is orthogonal to all lower degrees and has degree $n + 1$. For a real weight on a real interval, multiplication by x is symmetric. If $j \leq n - 2$, then

$$\langle q_j, xq_n \rangle_w = \langle xq_j, q_n \rangle_w = 0,$$

since xq_j has degree at most $n - 1$. Consequently multiplication by x produces only q_{n+1} , q_n , and q_{n-1} : this is the origin of the three-term recurrence for orthogonal polynomials.

Now the least-squares polynomial of degree at most N is simply

$$P_N f = \sum_{n=0}^N \langle q_n, f \rangle_w q_n.$$

Increasing N adds terms without changing earlier orthogonal coefficients. This is the practical advantage of an orthogonal basis.

11.3 Legendre polynomials as a worked construction

On $[-1, 1]$ with $w = 1$, Gram–Schmidt gives

$$q_0 = \frac{1}{\sqrt{2}}, \quad q_1 = \sqrt{\frac{3}{2}}x.$$

For x^2 , parity removes its projection onto q_1 , while the projection onto constants is $1/3$. Thus

$$r_2 = x^2 - \frac{1}{3}, \quad \|r_2\|^2 = \int_{-1}^1 \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9} \right) dx = \frac{8}{45}.$$

It follows that

$$q_2 = \sqrt{\frac{5}{2}} \frac{3x^2 - 1}{2}.$$

For x^3 , only its projection onto x survives:

$$r_3 = x^3 - \frac{\int_{-1}^1 x^4 dx}{\int_{-1}^1 x^2 dx} x = x^3 - \frac{3}{5}x, \quad \|r_3\|^2 = \frac{8}{175}.$$

Hence $q_3 = \sqrt{7/2}(5x^3 - 3x)/2$.

The conventional Legendre polynomials have $P_n(1) = 1$, rather than unit norm:

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{3x^2 - 1}{2}, \quad P_3 = \frac{5x^3 - 3x}{2}, \quad q_n = \sqrt{\frac{2n+1}{2}} P_n.$$

Their closed form is Rodrigues' formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad \int_{-1}^1 P_n P_m dx = \frac{2}{2n+1} \delta_{nm}. \quad (11.2)$$

The orthogonality follows immediately by integrating Rodrigues' formula against a polynomial Q_m of degree $m < n$ and integrating by parts n times. All boundary terms vanish because $(x^2 - 1)^n$ has zeros of order n at both endpoints. The remaining integral contains $Q_m^{(n)} = 0$. The Legendre differential equation and its angular applications are developed in Section 27.

12 Distributions and the identity kernel

12.1 Matrices become integral kernels

A linear functional maps a vector to a scalar, for example $v \mapsto \langle u, v \rangle$. A linear operator maps vectors to vectors. In a finite-dimensional basis,

$$g_i = \sum_j M_{ij} f_j.$$

The continuum analogue is an integral kernel:

$$(Mf)(x) = \int_a^b M(x, y) f(y) dy.$$

The two variables play the roles of row and column indices. Under composition, matrix multiplication becomes

$$(MN)(x, z) = \int_a^b M(x, y) N(y, z) dy.$$

For a weighted measure, use $w(y) dy$ consistently instead.

The identity matrix has entries δ_{ij} . Its continuum counterpart is the kernel $\delta(x - y)$, characterized by

$$\int_a^b \delta(x - y) f(y) dy = f(x), \quad a < x < b. \quad (12.1)$$

For continuous f , this is a pointwise statement. For general L^2 functions, the identity is understood in the weak sense or as an equality of equivalence classes. An arbitrary choice of value at a single point cannot be recovered from an L^2 vector.

12.2 Delta as a distribution

A distribution is specified by its action on smooth test functions. The Dirac delta centered at a acts by

$$\langle \delta_a, \varphi \rangle = \varphi(a).$$

Here the brackets denote a distribution acting on a test function, not a Hilbert-space inner product. The definition does not assign a numerical value to “ $\delta(0)$ ”. Describing delta merely as zero away from the origin and infinite at the origin does not determine its action and is not a definition of a function.

Useful approximations are the following families, interpreted through their integrals against test functions:

$$\delta_\epsilon^{\text{box}}(x) = \begin{cases} \epsilon^{-1}, & |x| < \epsilon/2, \\ 0, & |x| > \epsilon/2, \end{cases} \quad (12.2)$$

$$\delta_\epsilon^{\text{Lor}}(x) = \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}, \quad (12.3)$$

$$\delta_\epsilon^{\text{sinc}}(x) = \frac{\sin(x/\epsilon)}{\pi x}. \quad (12.4)$$

For the box approximation,

$$\int \delta_\epsilon^{\text{box}}(x) \varphi(x) dx = \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} \varphi(x) dx \longrightarrow \varphi(0)$$

by continuity. For the Lorentzian, substitute $x = \epsilon t$:

$$\int \delta_\epsilon^{\text{Lor}}(x) \varphi(x) dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(\epsilon t)}{1 + t^2} dt \longrightarrow \varphi(0),$$

since $\pi^{-1} \int (1+t^2)^{-1} dt = 1$ and dominated convergence applies to bounded test functions. The sinc representation follows from

$$\frac{1}{2\pi} \int_{-1/\epsilon}^{1/\epsilon} e^{ikx} dk = \frac{\sin(x/\epsilon)}{\pi x}.$$

It approaches delta by admitting an increasingly wide range of Fourier wave numbers. Unlike the first two examples, it oscillates and is not nonnegative.

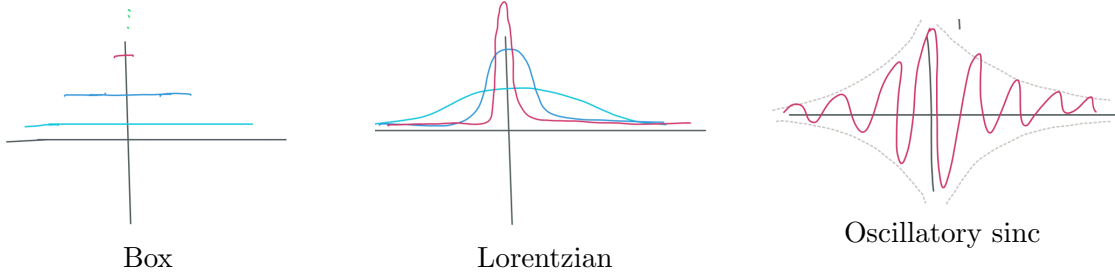


Figure 13: Three original sketches illustrating different distributional approximations to the same Dirac delta.

These approximations do not converge in L^2 to a delta function. For example,

$$\|\delta_\epsilon^{\text{box}}\|_{L^2}^2 = \epsilon \frac{1}{\epsilon^2} = \frac{1}{\epsilon} \longrightarrow \infty.$$

Delta is not a square-integrable function. Distributional convergence means convergence after integration against test functions; it is a different notion from norm convergence or pointwise convergence.

12.3 Completeness as a resolution of the identity

In finite dimensions, an orthonormal basis gives

$$I = \sum_n |e_n\rangle \langle e_n|, \quad Iv = \sum_n e_n \langle e_n, v \rangle = v.$$

For a complete orthonormal family u_n in $L_w^2(a, b)$, write its projection kernel as

$$K_N(x, y) = \sum_{n=1}^N u_n(x) u_n(y)^*.$$

Then

$$\int_a^b K_N(x, y) f(y) w(y) dy = \sum_{n=1}^N u_n(x) \langle u_n, f \rangle_w = S_N f(x).$$

Since $S_N f \rightarrow f$ in norm, the limiting kernel represents the identity:

$$\sum_n u_n(x) u_n(y)^* = \frac{\delta(x - y)}{w(y)}. \quad (12.5)$$

This equation is interpreted under an integral with measure $w(y) dy$. In the unweighted case, its right-hand side is simply $\delta(x - y)$. It is not a pointwise equality of ordinary functions, and the kernels themselves need not converge in L^2 .

For example, the sine basis gives

$$2 \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi y) = \delta(x - y) \quad (0 < x, y < 1).$$

For periodic functions of period one,

$$\sum_{n \in \mathbb{Z}} e^{2\pi i n(x-y)} = \sum_{m \in \mathbb{Z}} \delta(x - y - m).$$

The right-hand side is the periodic delta kernel. On one fundamental interval it acts as the identity on periodic functions.

12.4 Heaviside kernels and integration

Define

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Its value at zero is irrelevant to ordinary integrals. On a finite interval it is square integrable. The kernel $\Theta(x - y)$ implements integration from the left endpoint:

$$(Jf)(x) = \int_a^b \Theta(x - y) f(y) dy = \int_a^x f(y) dy. \quad (12.6)$$

For $f \in L^2(a, b)$, Cauchy–Schwarz gives

$$|Jf(x)| \leq \sqrt{x - a} \|f\|_{L^2},$$

so the integral is well-defined. It is an absolutely continuous function whose derivative equals f almost everywhere. If f is continuous, the fundamental theorem of calculus gives the derivative at every interior point.

Distributionally, $\Theta' = \delta$. In fact, for a compactly supported test function,

$$\langle \Theta', \varphi \rangle = - \int_0^{\infty} \varphi'(x) dx = \varphi(0).$$

Consequently $\partial_x \Theta(x - y) = \delta(x - y)$, and differentiating (12.6) recovers the identity operator.

12.5 Derivatives of delta and their domains

Distributional differentiation is defined by moving the derivative onto the test function:

$$\langle T^{(n)}, \varphi \rangle = (-1)^n \langle T, \varphi^{(n)} \rangle. \quad (12.7)$$

In particular,

$$\int \delta^{(n)}(y - a) \varphi(y) dy = (-1)^n \varphi^{(n)}(a).$$

The derivative kernel for an operator is often written differently:

$$\int \partial_x^n \delta(x - y) f(y) dy = f^{(n)}(x). \quad (12.8)$$

There is no extra minus sign in this form because $\partial_x \delta(x - y) = -\partial_y \delta(x - y)$. For $n = 1$,

$$\begin{aligned} \int \partial_x \delta(x - y) f(y) dy &= - \int \partial_y \delta(x - y) f(y) dy \\ &= \int \delta(x - y) f'(y) dy = f'(x), \end{aligned}$$

with vanishing endpoint terms for interior x or compactly supported functions.

For classical pointwise interpretation, there is a hierarchy:

$$\Theta(x - y) : f \mapsto \int_a^x f, \quad \delta(x - y) : f \mapsto f(x), \quad \partial_x^n \delta(x - y) : f \mapsto f^{(n)}(x).$$

The first operation makes sense for integrable functions, evaluation requires a specified point value, and the last requires the indicated classical derivative. In the weak formulation, derivatives can act on more general functions, but if the result must again belong to L^2 , we must restrict the operator domain accordingly. This leads naturally to the domain of a differential operator.

12.6 Scaling, dimensions, and more than one dimension

The normalization $\int \delta(x) dx = 1$ implies that delta has inverse units to its argument. If x is a length, $[\delta(x)] = L^{-1}$. For a nonzero constant a , a change of variable gives

$$\delta(ax) = \frac{1}{|a|} \delta(x).$$

More generally, if $g(x_j) = 0$ and all zeros are simple,

$$\delta(g(x)) = \sum_j \frac{\delta(x - x_j)}{|g'(x_j)|}.$$

This formula follows by using g as a local integration variable near each zero.

In d Cartesian dimensions,

$$\delta^{(d)}(\mathbf{x} - \mathbf{y}) = \prod_{j=1}^d \delta(x_j - y_j), \quad [\delta^{(d)}] = L^{-d}.$$

Under coordinates q with volume element $d^d x = J(q) d^d q$,

$$\delta^{(d)}(\mathbf{x}(q) - \mathbf{x}(q_0)) = \frac{\delta^{(d)}(q - q_0)}{|J(q_0)|},$$

provided the coordinate map is nonsingular at q_0 . For polar coordinates and $r_0 > 0$,

$$\delta^{(2)}(\mathbf{r} - \mathbf{r}_0) = \frac{1}{r_0} \delta(r - r_0) \delta_{2\pi}(\theta - \theta_0).$$

At the origin the angular coordinate is undefined. For radial integrals one may write

$$\delta^{(2)}(\mathbf{r}) = \frac{\delta_+(r)}{2\pi r}, \quad \int_0^\infty \delta_+(r) F(r) dr = F(0).$$

The notation δ_+ specifies a unit delta on the half-line and avoids ambiguity about an endpoint factor. The normalization is verified by integrating with the area element $r dr d\theta$.

Part III

Differential Operators

Differential
operators

13 Operator algebra and formal adjoints

13.1 The differential expression and its matrix analogy

An n th-order linear differential expression has the form

$$\mathcal{L} = \sum_{j=0}^n a_j(x) D^j, \quad D = \frac{d}{dx}, \quad a_n \neq 0. \quad (13.1)$$

The coefficients may depend on position and may be complex. Where a classical calculation is used, enough coefficient regularity and enough derivatives of the functions are assumed. A full operator requires more information: its Hilbert space, interval, weight, and domain.

The distribution kernel corresponding to (13.1) is

$$L(x, y) = \sum_{j=0}^n a_j(x) \partial_x^j \delta(x - y), \quad (\mathcal{L}f)(x) = \int L(x, y) f(y) dy.$$

Thus differential operators fit into the integral-kernel description. A finite-difference approximation makes the matrix analogy explicit. On a uniform grid with spacing h ,

$$f'(x_j) \approx \frac{f_{j+1} - f_{j-1}}{2h}, \quad -f''(x_j) \approx \frac{2f_j - f_{j-1} - f_{j+1}}{h^2}.$$

The derivative is represented by an antisymmetric tridiagonal matrix, while the negative second derivative has a symmetric tridiagonal matrix. The first and last rows depend on boundary conditions. Their role is the finite-dimensional counterpart of choosing a domain.

13.2 Noncommutativity: work on a test function

Multiplication and differentiation do not commute:

$$D(xf) = f + xf', \quad x(Df) = xf', \quad [D, x]f = f.$$

More generally, if M_a denotes multiplication by $a(x)$,

$$[D, M_a] = M_{a'}.$$

The safest way to multiply differential expressions is to apply them to a function and use the product rule before suppressing that function.

For example, let $L_1 = D + f(x)$ and $L_2 = D + g(x)$. For a trial function u ,

$$\begin{aligned} L_1 L_2 u &= (D + f)(u' + gu) \\ &= u'' + g'u + gu' + fu' + fgu \\ &= u'' + (f + g)u' + (g' + fg)u, \\ L_2 L_1 u &= u'' + (f + g)u' + (f' + fg)u. \end{aligned}$$

Hence

$$[L_1, L_2] = g' - f', \quad (13.2)$$

where the right-hand side is a multiplication operator. The expressions commute only if $g' - f' = 0$. For unbounded operators, the products must also be defined on a common domain; agreement of expressions alone does not settle the operator-domain question.

13.3 Hermitian matrices as the finite-dimensional model

For a matrix A , its adjoint is $A^\dagger = (A^*)^\top$, defined without coordinates by

$$\langle u, Av \rangle = \langle A^\dagger u, v \rangle.$$

A matrix is Hermitian if $A^\dagger = A$. If $Av = \lambda v$, then

$$\lambda \|v\|^2 = \langle v, Av \rangle = \langle Av, v \rangle = \lambda^* \|v\|^2,$$

so λ is real. For two eigenvectors,

$$(\lambda_n - \lambda_m) \langle v_m, v_n \rangle = 0.$$

Distinct eigenvalues therefore have orthogonal eigenvectors. Within a degenerate eigenspace, Gram-Schmidt produces an orthonormal basis without changing the eigenvalue.

A Hermitian $N \times N$ matrix has a complete orthonormal eigenbasis. Putting the eigenvectors into the columns of U gives

$$U^\dagger U = I, \quad U^\dagger A U = \text{diag}(\lambda_1, \dots, \lambda_N).$$

Thus U is unitary and diagonalizes A . A real orthogonal matrix is the real-valued special case of a unitary matrix. The differential-operator analogue requires an integration-by-parts identity and appropriate boundary conditions.

13.4 The formal adjoint from repeated integration by parts

Let the inner product be (10.1). Repeated integration by parts gives

$$\begin{aligned} \langle u, \mathcal{L}v \rangle_w &= \sum_{j=0}^n \int_a^b w u^* a_j v^{(j)} dx \\ &= \sum_{j=0}^n (-1)^j \int_a^b \frac{d^j}{dx^j} (w a_j u^*) v dx + B[u, v]. \end{aligned}$$

The first term is $\langle \mathcal{L}^{\text{f}\dagger} u, v \rangle_w$, where

$$\mathcal{L}^{\text{f}\dagger} u = \frac{1}{w} \sum_{j=0}^n (-1)^j \frac{d^j}{dx^j} (w a_j^* u). \quad (13.3)$$

The superscript $\text{f}\dagger$ emphasizes that this is the **formal adjoint**, before its domain has been determined. The difference of the two integrands is a total derivative:

$$w [u^* \mathcal{L} v - (\mathcal{L}^{\text{f}\dagger} u)^* v] = \frac{d}{dx} Q[u, v]. \quad (13.4)$$

This is Lagrange's identity. Its integrated boundary form is

$$B[u, v] = Q[u, v](b) - Q[u, v](a).$$

The expression Q is conjugate linear in u , linear in v , and involves endpoint derivatives through order at most $n - 1$.

Example 13.1 (Momentum expression). For $P = -iD$ and $w = 1$,

$$\begin{aligned} \langle u, Pv \rangle &= \int_a^b -iu^* v' dx \\ &= [-iu^* v]_a^b + \int_a^b i(u')^* v dx \\ &= \langle Pu, v \rangle - i[u^* v]_a^b. \end{aligned}$$

Therefore

$$P^{\text{f}\dagger} = -iD = P, \quad Q[u, v] = -iu^* v.$$

The expression is formally self-adjoint. Whether the operator is self-adjoint depends on what boundary values its domain allows.

13.5 Second-order expressions and Sturm–Liouville form

Consider

$$\mathcal{L} = a(x)D^2 + b(x)D + c(x)$$

first with unit weight. Formula (13.3) gives

$$\begin{aligned} \mathcal{L}^{\text{f}\dagger} u &= (a^* u)'' - (b^* u)' + c^* u \\ &= a^* u'' + (2a^{*'} - b^*)u' + (a^{*''} - b^{*'} + c^*)u. \end{aligned}$$

For real coefficients, formal self-adjointness requires $b = a'$, after which the zeroth-order condition is automatic. Thus

$$\mathcal{L} = (aD)' + c, \quad (aD)'u = (au')'.$$

The notation emphasizes that differentiation acts on both a and u' .

With a positive weight w , comparison of the first-derivative terms gives

$$(wa)' = wb.$$

For real coefficients on an interval where a has a fixed nonzero sign, this is a first-order equation for w :

$$\frac{w'}{w} = \frac{b - a'}{a}.$$

Integration yields

$$w(x) = \frac{C}{|a(x)|} \exp\left(\int^x \frac{b(s)}{a(s)} ds\right), \quad C > 0. \quad (13.5)$$

Indeed,

$$w\mathcal{L}u = (wa, u')' + wc, u.$$

Thus a nonsingular real second-order expression can be put in a formally self-adjoint form using a suitable weight. This construction is local when the leading coefficient vanishes or changes sign, and it does not by itself impose self-adjoint boundary conditions.

The standard sign convention for a Sturm–Liouville operator is

$$Au = \frac{1}{w(x)} [-(p(x)u')' + q(x)u], \quad p > 0, \quad w > 0, \quad q \text{ real}. \quad (13.6)$$

Its eigenvalue equation is

$$-(pu')' + qu = \lambda wu.$$

The opposite overall sign is also used in applications; the spectral parameter must be changed consistently. Direct calculation gives

$$\begin{aligned} w[u^*Av - (Au)^*v] &= -u^*(pv')' + (pu^*)'v \\ &= \frac{d}{dx} [p(u^*v - u^*v')]. \end{aligned}$$

Therefore

$$\langle u, Av \rangle_w - \langle Au, v \rangle_w = [p(u^*v - u^*v')]_a^b. \quad (13.7)$$

The weight disappears from the boundary form but remains in the inner product and eigenvalue problem.

14 Domains and genuinely self-adjoint operators

14.1 Why the differential expression is not enough

For a bounded matrix, every vector is an allowed input. Differentiation is different: most L^2 functions do not have an L^2 derivative. We must specify a domain $\mathcal{D}(A) \subset \mathcal{H}$ on which both u and Au belong to the Hilbert space.

For the examples below, $H^1(a, b)$ denotes functions with one weak derivative in L^2 , and $H^2(a, b)$ denotes functions with two. In one dimension, these regularity conditions give well-defined endpoint traces of u , and for H^2 also of u' . They provide convenient precise domains for first- and second-order expressions.

A regular n th-order boundary-value problem usually uses n independent boundary conditions involving derivatives of order below n . The domain of a linear operator must be a vector space, so the boundary conditions must be homogeneous and linear. For a second-order expression, the regular boundary conditions can be written

$$B_j[u] = \alpha_{j1}u(a) + \alpha_{j2}u'(a) + \beta_{j1}u(b) + \beta_{j2}u'(b) = 0, \quad j = 1, 2. \quad (14.1)$$

These equations may connect the endpoints rather than prescribe each endpoint separately. Merely choosing the right number of conditions does not guarantee self-adjointness.

Nonhomogeneous conditions, such as $u(a) = A \neq 0$, form an affine set, not a vector space. In an inhomogeneous problem one may subtract a known function satisfying the boundary data and solve for a remaining function with homogeneous conditions.

The domains considered here are dense in L^2 . For example, smooth functions supported strictly inside (a, b) satisfy homogeneous endpoint conditions and are dense in $L^2(a, b)$. Density ensures uniqueness of the adjoint operator.

14.2 The adjoint domain

For a densely defined operator A , a vector u belongs to $\mathcal{D}(A^\dagger)$ when there exists $h \in \mathcal{H}$ such that

$$\langle u, Av \rangle = \langle h, v \rangle \quad \text{for every } v \in \mathcal{D}(A).$$

The vector h is unique, and $A^\dagger u = h$. For regular differential operators, integration by parts first identifies the formal adjoint expression. Then the boundary form must vanish for every allowed v ; this determines the boundary conditions on u .

Definition 14.1. An operator is **symmetric** if

$$\langle u, Av \rangle = \langle Au, v \rangle \quad (u, v \in \mathcal{D}(A)).$$

It is **self-adjoint** if

$$A^\dagger = A, \quad \mathcal{D}(A^\dagger) = \mathcal{D}(A).$$

For a symmetric operator, $\mathcal{D}(A) \subseteq \mathcal{D}(A^\dagger)$ and the two operators agree on $\mathcal{D}(A)$. Equality of domains is the additional requirement. Checking the boundary form only for pairs already in $\mathcal{D}(A)$ proves symmetry, but may fail to detect a larger adjoint domain.

14.3 Dirichlet second derivative: determination of both domains

Take

$$A = -D^2, \quad \mathcal{D}(A) = \{v \in H^2(a, b) : v(a) = v(b) = 0\}.$$

Its formal adjoint is itself. From (13.7),

$$\langle u, Av \rangle - \langle Au, v \rangle = [u^*v' - u'^*v]_a^b.$$

Since $v(a) = v(b) = 0$, the right-hand side reduces to

$$-u(b)^*v'(b) + u(a)^*v'(a).$$

The values $v'(a)$ and $v'(b)$ are independently variable in the domain. For this expression to vanish for every v , we must have $u(a) = u(b) = 0$. Thus the adjoint has exactly the same boundary conditions,

$$\mathcal{D}(A^\dagger) = \{u \in H^2(a, b) : u(a) = u(b) = 0\} = \mathcal{D}(A).$$

This proves self-adjointness, rather than only formal self-adjointness.

14.4 Momentum with two zero endpoint values

Now use the same-looking endpoint conditions for a first-order operator:

$$P = -iD, \quad \mathcal{D}(P) = \{v \in H^1(a, b) : v(a) = v(b) = 0\}.$$

The boundary form is $-i[u^*v]_a^b$. It vanishes for every $u, v \in \mathcal{D}(P)$, so P is symmetric. But when finding the adjoint domain, $v(a) = v(b) = 0$ already makes the boundary form vanish for every $u \in H^1(a, b)$, without imposing any endpoint conditions on u . Therefore

$$\mathcal{D}(P^\dagger) = H^1(a, b) \supsetneq \mathcal{D}(P).$$

The operator is not self-adjoint.

The eigenvalue problem exposes the failure of the expected basis:

$$-iu' = \lambda u \implies u(x) = Ce^{i\lambda(x-a)}.$$

The condition $u(a) = 0$ forces $C = 0$. There are no nonzero eigenfunctions satisfying the two endpoint conditions. Formal self-adjointness and a vanishing boundary form on the proposed domain were insufficient.

14.5 Momentum with only one zero endpoint value

Taking a single condition also fails to make momentum self-adjoint. For example,

$$\mathcal{D}(P_{\text{left}}) = \{v \in H^1(a, b) : v(a) = 0\}.$$

Now the boundary form becomes $-iu(b)^*v(b)$. To make it vanish for all v , the adjoint requires $u(b) = 0$:

$$\mathcal{D}(P_{\text{left}}^\dagger) = \{u \in H^1(a, b) : u(b) = 0\}.$$

The operator and its adjoint impose conditions at opposite endpoints, so their domains differ. In fact, the one-endpoint operator is not even symmetric: for $u = v$ in its domain, the boundary form need not vanish. Reversing the endpoints interchanges the two domains.

14.6 Twisted periodic boundary conditions

A self-adjoint momentum domain connects the two endpoints. Let

$$\mathcal{D}(P_\kappa) = \{v \in H^1(a, b) : v(b) = \kappa v(a)\}, \quad \kappa \neq 0.$$

For u, v both satisfying this condition,

$$[u^*v]_a^b = (|\kappa|^2 - 1)u(a)^*v(a).$$

Symmetry therefore requires $|\kappa| = 1$. Write $\kappa = e^{i\theta}$, with real θ modulo 2π . To verify equality of domains, allow u to lie in the adjoint domain and use only $v(b) = e^{i\theta}v(a)$:

$$[u^*v]_a^b = (e^{i\theta}u(b)^* - u(a)^*)v(a).$$

Since $v(a)$ is arbitrary, this vanishes exactly when $u(b) = e^{i\theta}u(a)$. Thus

$$P_\theta = -iD, \quad \mathcal{D}(P_\theta) = \{u \in H^1(a, b) : u(b) = e^{i\theta}u(a)\} \quad (14.2)$$

is self-adjoint. These are the **twisted periodic** boundary conditions. The special cases $\theta = 0$ and $\theta = \pi$ are periodic and antiperiodic, respectively.

Set $L = b - a$. Solving $-iu' = \lambda u$ and imposing the boundary condition gives

$$e^{i\lambda L} = e^{i\theta}, \quad \lambda_n = \frac{2\pi n + \theta}{L}, \quad n \in \mathbb{Z}.$$

The normalized eigenfunctions are

$$u_n(x) = \frac{1}{\sqrt{L}} \exp\left[\frac{i(2\pi n + \theta)(x - a)}{L}\right]. \quad (14.3)$$

Their orthogonality can be checked without approximation:

$$\langle u_m, u_n \rangle = \frac{1}{L} \int_a^b \exp\left[\frac{2\pi i(n - m)(x - a)}{L}\right] dx = \delta_{mn}.$$

Multiplication by $e^{i\theta(x-a)/L}$ is unitary and converts the ordinary Fourier basis into this basis. Completeness therefore follows from Fourier completeness. The phase shifts all eigenvalues while preserving their spacing $2\pi/L$.

14.7 The half-line obstruction for momentum

On $[0, \infty)$, consider $P = -iD$. For functions in $H^1(0, \infty)$, the continuous representative tends to zero at infinity. One useful estimate follows from

$$|u(x)|^2 = - \int_x^\infty \frac{d}{ds} |u(s)|^2 ds,$$

first justified by approximation, giving

$$|u(x)|^2 \leq 2 \|u\|_{L^2(x, \infty)} \|u'\|_{L^2(x, \infty)} \longrightarrow 0.$$

The boundary form consequently reduces to

$$\langle u, Pv \rangle - \langle Pu, v \rangle = iu(0)^*v(0).$$

For a symmetric domain, taking $u = v$ forces $u(0) = 0$. But then the adjoint boundary form vanishes for every $u \in H^1(0, \infty)$, since $v(0) = 0$. Thus the natural symmetric domain has a strictly larger adjoint domain. There is no second finite endpoint with which to pair the boundary value by a phase.

The obstruction can also be measured by the deficiency equations for the adjoint:

$$P^\dagger u = +iu \implies u = Ce^{-x}, \quad P^\dagger u = -iu \implies u = Ce^x.$$

Only the first solution is square integrable. The two deficiency spaces have dimensions 1 and 0, so the symmetric half-line momentum operator has no self-adjoint extension. This does not prevent the second-order operator $-D^2$ on the half-line from having self-adjoint Dirichlet, Neumann, or appropriate Robin realizations.

15 Sturm–Liouville eigenfunction expansions

15.1 Regular self-adjoint boundary conditions

For the regular expression (13.6) on a finite interval, separated real boundary conditions have the form

$$\alpha_a u(a) + \beta_a p(a)u'(a) = 0, \quad \alpha_b u(b) + \beta_b p(b)u'(b) = 0,$$

with real coefficients, not both zero at either endpoint. If $\beta_a \neq 0$, then $p(a)u'(a) = -(\alpha_a/\beta_a)u(a)$, and the endpoint contribution to (13.7) vanishes because the proportionality factor is real. If $\beta_a = 0$, both functions vanish at that endpoint, with the same result. The argument at b is identical.

These conditions include Dirichlet ($u = 0$), Neumann ($pu' = 0$), and Robin (real linear combinations of the two). Coupled conditions are also possible. For instance,

$$u(b) = e^{i\theta}u(a), \quad p(b)u'(b) = e^{i\theta}p(a)u'(a)$$

make the endpoint terms cancel and give a twisted periodic second-order problem.

15.2 Reality and orthogonality: full proofs

Suppose $Au_n = \lambda_n u_n$ and the boundary conditions make A self-adjoint. Then

$$\lambda_n \|u_n\|_w^2 = \langle u_n, Au_n \rangle_w = \langle Au_n, u_n \rangle_w = \lambda_n^* \|u_n\|_w^2.$$

Since an eigenfunction is nonzero, $\lambda_n = \lambda_n^*$. For two eigenfunctions,

$$\begin{aligned} \lambda_n \langle u_m, u_n \rangle_w &= \langle u_m, Au_n \rangle_w \\ &= \langle Au_m, u_n \rangle_w = \lambda_m \langle u_m, u_n \rangle_w. \end{aligned}$$

Therefore

$$(\lambda_n - \lambda_m) \langle u_m, u_n \rangle_w = 0. \quad (15.1)$$

Distinct eigenvalues imply orthogonality. If an eigenvalue is degenerate, any linear combination of its eigenfunctions remains an eigenfunction, so Gram–Schmidt may be applied within that eigenspace.

For separated regular Sturm–Liouville conditions, eigenvalues are simple. If u, v have the same eigenvalue, subtraction of their equations gives

$$\frac{d}{dx} [p(uv' - u'v)] = 0.$$

The separated boundary condition at one endpoint forces this constant Wronskian to vanish. Uniqueness of the second-order initial-value problem then implies that u and v are proportional. Periodic problems can instead have degeneracies, as the sine and cosine modes show.

15.3 Completeness and what it does not say

For a regular self-adjoint Sturm–Liouville problem on a finite interval with $p, w > 0$, the spectrum is discrete, the eigenvalues have no finite accumulation point, and normalized eigenfunctions form a complete orthonormal basis of L_w^2 . They may be ordered so that $\lambda_n \rightarrow +\infty$ for the sign convention (13.6). Completeness is a separate infinite-dimensional spectral theorem; the short proofs of reality and orthogonality do not prove it.

Consequently, for every $f \in L_w^2$,

$$f = \sum_n c_n u_n, \quad c_n = \int_a^b u_n(x)^* f(x) w(x) dx, \quad (15.2)$$

$$\|f\|_w^2 = \sum_n |c_n|^2, \quad \sum_n u_n(x) u_n(y)^* = \frac{\delta(x-y)}{w(y)}. \quad (15.3)$$

The function being expanded need not itself satisfy the operator's boundary conditions, or even belong to its domain. The expansion converges in L_w^2 ; it need not preserve endpoint values in a pointwise limit. Applying A term by term is an additional operation. In the self-adjoint eigenbasis, its domain is characterized by

$$\sum_n |\lambda_n|^2 |c_n|^2 < \infty, \quad Af = \sum_n \lambda_n c_n u_n \quad \text{in } L_w^2.$$

Thus convergence of f does not automatically imply convergence of its derivative or of Af .

15.4 The Dirichlet interval: solve every eigenvalue case

For

$$-u'' = \lambda u, \quad u(0) = u(L) = 0,$$

first suppose $\lambda = k^2 > 0$. Then

$$u(x) = A \cos kx + B \sin kx.$$

The condition at zero gives $A = 0$, and the condition at L gives $B \sin kL = 0$. A nonzero solution requires $kL = n\pi$ with $n \geq 1$. For $\lambda = 0$, the solution $A + Bx$ vanishes at both endpoints only if $A = B = 0$. For $\lambda = -\kappa^2 < 0$, the solution is $A \cosh \kappa x + B \sinh \kappa x$; again the endpoint conditions force both coefficients to vanish. Hence the complete list is

$$\lambda_n = \frac{n^2 \pi^2}{L^2}, \quad u_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \quad n = 1, 2, \dots \quad (15.4)$$

The normalization follows from $\int_0^L \sin^2(n\pi x/L) dx = L/2$.

Positivity can also be seen directly from integration by parts:

$$\langle u, Au \rangle = \int_0^L u^* (-u'') dx = \int_0^L |u'|^2 dx \geq 0,$$

because the endpoint term vanishes. Zero would require a constant function, and Dirichlet conditions then force that function to be zero.

Example 15.1 (Expansion of $x(L-x)$). Let $f(x) = x(L-x)$. Write a sine series with unnormalized sines, $f(x) = \sum_{n \geq 1} b_n \sin(n\pi x/L)$, where

$$b_n = \frac{2}{L} \int_0^L x(L-x) \sin(k_n x) dx, \quad k_n = \frac{n\pi}{L}.$$

Integrate by parts twice. The first boundary term vanishes since $f(0) = f(L) = 0$. The next boundary term contains $\sin(k_n x)$ and also vanishes. Since $f'' = -2$,

$$\begin{aligned} \int_0^L f \sin(k_n x) dx &= -\frac{1}{k_n^2} \int_0^L f'' \sin(k_n x) dx \\ &= \frac{2}{k_n^2} \frac{1 - (-1)^n}{k_n}. \end{aligned}$$

Therefore

$$b_n = \frac{4L^2}{n^3 \pi^3} (1 - (-1)^n),$$

and

$$x(L-x) = \frac{8L^2}{\pi^3} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{1}{n^3} \sin \frac{n\pi x}{L}.$$

The extra decay n^{-3} , compared with the constant-function example, reflects the compatibility of f with the endpoint conditions and its additional smoothness.

15.5 Neumann, mixed, and periodic interval spectra

The same differential expression produces different spectra when its domain changes. For Neumann conditions $u'(0) = u'(L) = 0$, the normalized modes are

$$u_0 = L^{-1/2}, \quad \lambda_0 = 0; \quad u_n = \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}, \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n \geq 1.$$

The constant zero mode is part of the basis and cannot be discarded. For mixed conditions $u(0) = 0, u'(L) = 0$,

$$u_n = \sqrt{\frac{2}{L}} \sin \frac{(n + \frac{1}{2})\pi x}{L}, \quad \lambda_n = \frac{(n + \frac{1}{2})^2\pi^2}{L^2}, \quad n \geq 0.$$

The half-integer wave numbers follow from $\cos(kL) = 0$.

For periodic conditions on $[-L/2, L/2]$,

$$u(-L/2) = u(L/2), \quad u'(-L/2) = u'(L/2),$$

the normalized eigenfunctions and eigenvalues are

$$u_n(x) = \frac{1}{\sqrt{L}} e^{ik_n x}, \quad k_n = \frac{2\pi n}{L}, \quad \lambda_n = k_n^2, \quad n \in \mathbb{Z}.$$

For $n \neq 0$, n and $-n$ have the same eigenvalue. Equivalently, one may use real sine and cosine modes in each two-dimensional eigenspace. These examples illustrate why the operator must be specified together with its domain.

16 Discrete and continuous spectra; density of states

16.1 An infinite interval does not determine the spectral type

Passing from a finite interval to the full line does not always produce a continuous spectrum. Confinement can keep eigenfunctions localized and normalizable. Conversely, freely propagating modes need not be square integrable on the full line. Both situations occur in the same Hilbert space $L^2(\mathbb{R})$, depending on the operator.

Example 16.1 (A confining harmonic-oscillator operator). Consider

$$H = -D^2 + x^2$$

with its self-adjoint realization on $L^2(\mathbb{R})$. Introduce

$$a = \frac{D + x}{\sqrt{2}}, \quad a^\dagger = \frac{-D + x}{\sqrt{2}}.$$

Using $Dx = xD + 1$ gives

$$a^\dagger a = \frac{1}{2}(-D^2 + x^2 - 1), \quad [a, a^\dagger] = 1, \quad H = 2a^\dagger a + 1.$$

The ground-state equation $au_0 = 0$ becomes $u_0' + xu_0 = 0$, so

$$u_0(x) = \pi^{-1/4} e^{-x^2/2}, \quad Hu_0 = u_0.$$

Since $[H, a^\dagger] = 2a^\dagger$, repeated application raises the eigenvalue by two:

$$u_n = \frac{(a^\dagger)^n}{\sqrt{n!}} u_0, \quad \lambda_n = 2n + 1.$$

These functions are Hermite polynomials times a Gaussian,

$$u_n(x) = \frac{H_n(x) e^{-x^2/2}}{\pi^{1/4} \sqrt{2^n n!}},$$

so each is square integrable without a box-size normalization. The spectrum remains discrete even though the coordinate interval is infinite.

16.2 Free modes and box normalization

For $A = -D^2$ on the full line, formal eigenfunctions are e^{ikx} with eigenvalue k^2 . Their squared modulus is one, so

$$\int_{-\infty}^{\infty} |e^{ikx}|^2 dx = \infty.$$

They are generalized eigenfunctions, not elements of $L^2(\mathbb{R})$. Normalizable states are wave packets, which superpose a range of wave numbers.

To derive the continuum normalization, first put the system in a periodic box of length L . The wave numbers and modes are

$$k_n = \frac{2\pi n}{L}, \quad \Delta k = \frac{2\pi}{L}, \quad u_n(x) = \frac{1}{\sqrt{L}} e^{ik_n x}.$$

The number of modes in a wave-number interval dk is

$$dn = \frac{L}{2\pi} dk, \quad \frac{dn}{dk} = \frac{L}{2\pi}. \quad (16.1)$$

For functions whose variation across adjacent k_n becomes small as $L \rightarrow \infty$, the sum is a Riemann sum:

$$\sum_n F(k_n) \longrightarrow \frac{L}{2\pi} \int_{-\infty}^{\infty} F(k) dk.$$

The density per unit spatial length is $\rho_k = 1/(2\pi)$ when signed wave numbers are counted. If only $k > 0$ is used and both propagation directions are included, the density is twice as large.

16.3 Continuum completeness and Fourier inversion

The periodic-box completeness relation is

$$\sum_n u_n(x) u_n(y)^* = \frac{1}{L} \sum_n e^{ik_n(x-y)}.$$

Apply the sum-to-integral rule:

$$\begin{aligned} \frac{1}{L} \sum_n e^{ik_n(x-y)} &\longrightarrow \frac{1}{L} \frac{L}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} dk = \delta(x-y). \end{aligned}$$

The divergent number of modes and the vanishing factor $1/L$ cancel. The limiting statement is distributional.

Now expand a sufficiently regular localized f in box modes:

$$f(x) = \sum_n c_n u_n(x), \quad c_n = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} f(y) e^{-ik_n y} dy.$$

Define

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(y) e^{-iky} dy.$$

For large L , $c_n \simeq \tilde{f}(k_n)/\sqrt{L}$. Therefore

$$\begin{aligned} f(x) &\simeq \frac{1}{L} \sum_n \tilde{f}(k_n) e^{ik_n x} \\ &\longrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk. \end{aligned}$$

This derives the Fourier-transform pair, including its normalization, from a discrete orthonormal eigenfunction expansion:

$$\boxed{\tilde{f}(k) = \int_{\mathbb{R}} f(x) e^{-ikx} dx, \quad f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{f}(k) e^{ikx} dk.} \quad (16.2)$$

For general L^2 functions, the transform is defined by extension in norm rather than requiring both integrals to converge absolutely.

16.4 Delta-normalized eigenfunctions and Parseval

An equivalent symmetric convention uses

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}.$$

Then

$$\begin{aligned}\int_{\mathbb{R}} \phi_k(x)^* \phi_{k'}(x) dx &= \delta(k - k'), \\ \int_{\mathbb{R}} \phi_k(x) \phi_k(y)^* dk &= \delta(x - y).\end{aligned}$$

The first relation is delta normalization, not unit normalization in L^2 . If $a(k) = \tilde{f}(k)/\sqrt{2\pi}$, then

$$f(x) = \int_{\mathbb{R}} a(k) \phi_k(x) dk, \quad \|f\|_{L^2}^2 = \int_{\mathbb{R}} |a(k)|^2 dk = \frac{1}{2\pi} \int_{\mathbb{R}} |\tilde{f}(k)|^2 dk.$$

The last equality is the continuum form of Parseval. It also follows by substituting $c_n \simeq \tilde{f}(k_n)/\sqrt{L}$ into $\sum_n |c_n|^2$ and taking the box limit.

16.5 Density in wave number versus density in eigenvalue

A constant density in k does not imply a constant density in $\lambda = k^2$. Counting both signs of k , the number of states with $0 < k^2 \leq \lambda$ in a long periodic box is approximately

$$N(\lambda) \simeq \frac{L}{2\pi} (2\sqrt{\lambda}) = \frac{L\sqrt{\lambda}}{\pi}.$$

Differentiating gives

$$\rho_\lambda(\lambda) = \frac{dN}{d\lambda} = \frac{L}{2\pi\sqrt{\lambda}}, \quad \lambda > 0. \quad (16.3)$$

For a free quantum particle with $E = \hbar^2 k^2 / (2m)$, the change of variable gives

$$\rho_E(E) = \frac{L}{\pi\hbar} \sqrt{\frac{m}{2E}}.$$

These formulas count spatial modes without an additional spin or internal degeneracy. Such multiplicities must be included separately.

16.6 Mixed spectra and scattering

A potential can support localized bound states and propagating scattering states in the same problem. For a self-adjoint Schrödinger-type operator

$$H = -D^2 + V(x),$$

a mixed spectral expansion has the schematic form

$$f(x) = \sum_b c_b u_b(x) + \sum_\alpha \int_0^\infty c_\alpha(k) u_{k,\alpha}(x) dk, \quad (16.4)$$

$$\delta(x - y) = \sum_b u_b(x) u_b(y)^* + \sum_\alpha \int_0^\infty u_{k,\alpha}(x) u_{k,\alpha}(y)^* dk. \quad (16.5)$$

Here bound states have unit norm, continuum states have delta normalization, and α labels independent channels, such as left- and right-moving scattering states or even and odd parity. If a different normalization is used for the continuum functions, the integral includes the corresponding density-of-states factor. One must not insert both a density factor and a normalization that has already absorbed it.

Example 16.2 (An attractive delta well). Take $V(x) = -g\delta(x)$ with $g > 0$. Seek a bound state of eigenvalue $\lambda = -\kappa^2$. Away from the origin, normalizability gives

$$u(x) = Ce^{-\kappa|x|}.$$

Integrating the eigenvalue equation across zero gives the derivative jump

$$u'(0^+) - u'(0^-) = -g u(0).$$

For the trial function the left side is $-2\kappa C$, so $\kappa = g/2$ and $\lambda_b = -g^2/4$. Normalization gives

$$1 = |C|^2 \int_{\mathbb{R}} e^{-2\kappa|x|} dx = \frac{|C|^2}{\kappa}, \quad u_b(x) = \sqrt{\kappa} e^{-\kappa|x|}.$$

Positive eigenvalues k^2 describe extended scattering states. The bound state must be included along with the continuum to obtain completeness.

16.7 Two barriers, resonances, and redistribution of states

The lecture's barrier example is

$$V(x) = g [\delta(x - a) + \delta(x + a)], \quad g > 0.$$

At each barrier the wave function is continuous and its derivative jumps:

$$u'(a^+) - u'(a^-) = g u(a), \quad u'(-a^+) - u'(-a^-) = g u(-a).$$

These conditions follow by integrating $-u'' + Vu = k^2u$ over a shrinking interval containing the barrier. They connect the plane-wave solutions in the left, middle, and right regions.

For large but finite g , waves can spend a long time in the middle region and leak slowly through the barriers. Near the standing-wave frequencies of that region, the probability density in the middle becomes large. These are **resonances**; at finite barrier strength they are not square-integrable positive-energy eigenstates on the full line. Their finite lifetime produces a finite resonance width.

In the impenetrable limit $g \rightarrow \infty$, the interval $(-a, a)$ separates from the two exterior half-lines. The interior obeys Dirichlet conditions at $\pm a$ and has modes

$$u_n(x) = \frac{1}{\sqrt{a}} \sin \frac{n\pi(x+a)}{2a}, \quad k_n = \frac{n\pi}{2a}, \quad n \geq 1,$$

supported in the interior. The exterior regions retain continuous spectra. The limiting operator is a direct sum of a discrete interior problem and two continuous exterior problems, so both contributions are required in the completeness relation.

A large-box quantization condition makes the relation between scattering and density of states explicit. For one channel, let

$$kL_{\text{eff}} + \delta(k) = n\pi,$$

where $\delta(k)$ is the scattering phase in that channel and L_{eff} is the relevant free propagation length. Differentiating,

$$\frac{dn}{dk} = \frac{L_{\text{eff}}}{\pi} + \frac{1}{\pi} \frac{d\delta}{dk}. \quad (16.6)$$

Rapid phase variation near a resonance produces a peak in the density of states. As barriers become impenetrable, narrow resonances approach the isolated interior levels, while the exterior continuum has the available exterior length in its free density. This is a redistribution of spectral weight. A finite-box regularization provides a meaningful way to count states; an unregulated integral of the full continuum density is infinite.

Part IV

Linear Response and Green's Functions

Linear
response

17 Inhomogeneous equations and the Fredholm alternative

An eigenvalue problem describes the natural modes of an operator. An inhomogeneous problem asks how the same system responds to a specified source:

$$\mathcal{L}u = f. \quad (17.1)$$

The differential expression alone does not specify this problem. We must also choose its domain, including its boundary conditions. These conditions determine whether a solution exists, whether it is unique, and which Green's function is appropriate.

17.1 The finite-dimensional model

First consider the scalar equation $mx = a$. If $m \neq 0$, there is exactly one solution, $x = a/m$. If $m = 0$, the equation has no solution when $a \neq 0$, and every x is a solution when $a = 0$. The matrix equation $M\mathbf{x} = \mathbf{a}$ has a similar structure, but the condition on the source is now an orthogonality condition.

Theorem 17.1 (Finite-dimensional Fredholm alternative). *Let M be an $n \times n$ complex matrix. Then*

$$\text{ran } M = (\ker M^\dagger)^\perp. \quad (17.2)$$

Consequently, $M\mathbf{x} = \mathbf{a}$ is solvable if and only if

$$\mathbf{y}^\dagger \mathbf{a} = 0 \quad \text{for every } \mathbf{y} \in \ker M^\dagger. \quad (17.3)$$

When a solution $\mathbf{x}_0$ exists, all solutions are

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{z}, \quad \mathbf{z} \in \ker M. \quad (17.4)$$

Proof. If $\mathbf{a} = M\mathbf{x}$ and $M^\dagger \mathbf{y} = 0$, then

$$\mathbf{y}^\dagger \mathbf{a} = \mathbf{y}^\dagger M\mathbf{x} = (M^\dagger \mathbf{y})^\dagger \mathbf{x} = 0.$$

Thus $\text{ran } M \subseteq (\ker M^\dagger)^\perp$. Since $\text{rank } M = \text{rank } M^\dagger$, rank-nullity gives

$$\dim(\ker M^\dagger)^\perp = n - \dim \ker M^\dagger = \text{rank } M = \dim \text{ran } M.$$

The inclusion is therefore equality. Finally, if both $\mathbf{x}$ and $\mathbf{x}_0$ solve the same equation, then $M(\mathbf{x} - \mathbf{x}_0) = 0$. $\square$

For a square matrix, $\det M \neq 0$ is equivalent to $\ker M = \{0\}$, and the unique solution is $M^{-1}\mathbf{a}$. If $\det M = 0$, the nonzero vectors in $\ker M$ are called **zero modes**. There are $\dim \ker M = \dim \ker M^\dagger$ independent zero modes. A compatible source then gives infinitely many solutions; an incompatible source gives none.

17.2 Differential operators and compatibility

For a differential operator, the adjoint includes its adjoint boundary conditions. For regular boundary-value problems with the Fredholm property, the same alternative applies:

$$\mathcal{L}u = f \iff \langle v, f \rangle = 0 \text{ for every } v \in \ker \mathcal{L}^\dagger. \quad (17.5)$$

Here the equivalence refers to existence of a solution, and the inner product is the one appropriate to the function space. If a solution exists, its nonuniqueness is precisely the freedom to add an element of $\ker \mathcal{L}$. For the regular self-adjoint problems considered here, absence of a zero eigenvalue gives an inverse on the allowed sources. This conclusion requires the boundary-value assumptions; absence of an eigenvector at zero alone is not a general invertibility criterion for every infinite-dimensional operator.

Example 17.2 (Neumann boundary conditions). Consider

$$-u''(x) = f(x), \quad 0 < x < 1, \quad u'(0) = u'(1) = 0. \quad (17.6)$$

The operator $\mathcal{L} = -d^2/dx^2$ is self-adjoint on this domain, and $u_0(x) = 1$ is a zero mode. The compatibility condition is therefore

$$\int_0^1 f(x) dx = 0. \quad (17.7)$$

One can also see this directly by integrating the equation:

$$\int_0^1 f(x) dx = -u'(1) + u'(0) = 0.$$

Conversely, if the integral vanishes, then

$$u(x) = C - \int_0^x (x-s)f(s) ds \quad (17.8)$$

satisfies the equation and both boundary conditions. The arbitrary constant is the zero-mode freedom. Prescribing the mean $\int_0^1 u(x) dx$, for example, removes this freedom.

17.3 Nonhomogeneous boundary conditions

An inverse operator is most naturally defined on a linear domain with homogeneous boundary conditions. Nonzero boundary data can be handled by **lifting** them. For example, if $u(a) = A$ and $u(b) = B$, choose

$$h(x) = A \frac{b-x}{b-a} + B \frac{x-a}{b-a}, \quad u = h + v.$$

Then $v(a) = v(b) = 0$ and

$$\mathcal{L}v = f - \mathcal{L}h. \quad (17.9)$$

A Green's function for the homogeneous problem can now be applied to the modified source. If zero modes are present, compatibility must be checked for $f - \mathcal{L}h$, not for f alone.

18 Green's operators and spectral representations

18.1 A kernel for the inverse

Suppose first that $\mathcal{L}$ is invertible and the integration measure is ds . Its inverse can be represented by a kernel:

$$u(x) = (\mathcal{L}^{-1}f)(x) = \int_a^b G(x, s)f(s) ds. \quad (18.1)$$

The **Green's function** satisfies

$$\mathcal{L}_x G(x, s) = \delta(x - s), \quad (18.2)$$

and, as a function of x , satisfies the boundary conditions defining the problem. The subscript x means that all derivatives act on x , while s is held fixed. Applying $\mathcal{L}$ to (18.1) gives

$$\mathcal{L}u(x) = \int_a^b [\mathcal{L}_x G(x, s)]f(s) ds = \int_a^b \delta(x - s)f(s) ds = f(x).$$

These identities are interpreted distributionally when derivatives of the kernel are singular. The inverse is an operator; $G(x, s)$ is its integral kernel. A kernel need not itself be singular as a function: for a regular second-order problem it is often continuous, with a jump in its first derivative.

The physical interpretation is superposition. Resolve the source into localized sources,

$$f(x) = \int_a^b \delta(x - s)f(s) ds.$$

The response to a unit source at s is $G(x, s)$; adding the responses with weights $f(s) ds$ produces the response to the full source.

18.2 Weighted eigenfunction expansion

Let $\mathcal{L}$ be self-adjoint in the weighted space with inner product

$$\langle u, v \rangle_w = \int_a^b u(x)^* v(x) w(x) dx, \quad w(x) > 0.$$

Let $\{e_n\}$ be a complete orthonormal set of eigenfunctions,

$$\mathcal{L}e_n = \lambda_n e_n, \quad \int_a^b e_m^* e_n w dx = \delta_{mn}. \quad (18.3)$$

Expand both the solution and the source:

$$u = \sum_n u_n e_n, \quad f = \sum_n f_n e_n, \quad f_n = \langle e_n, f \rangle_w.$$

The equation reduces to the scalar equations

$$\lambda_n u_n = f_n. \quad (18.4)$$

When all $\lambda_n \neq 0$, substitute $u_n = f_n/\lambda_n$ and collect the integrals:

$$\begin{aligned} u(x) &= \sum_n \frac{e_n(x)}{\lambda_n} \int_a^b e_n(s)^* f(s) w(s) ds \\ &= \int_a^b G(x, s) f(s) w(s) ds, \end{aligned} \quad (18.5)$$

$$G(x, s) = \sum_n \frac{e_n(x) e_n(s)^*}{\lambda_n}. \quad (18.6)$$

The completeness relation is

$$\sum_n e_n(x) e_n(s)^* = \frac{\delta(x - s)}{w(s)}. \quad (18.7)$$

It follows immediately that

$$\mathcal{L}_x G(x, s) = \frac{\delta(x - s)}{w(s)}. \quad (18.8)$$

The weight must be retained consistently in both the kernel equation and the integral representation.

Since the eigenvalues of a self-adjoint operator are real,

$$G(x, s) = G(s, x)^*. \quad (18.9)$$

For a real differential expression with real boundary conditions, the Green's function can be chosen real, and this becomes $G(x, s) = G(s, x)$. This symmetry is also the kernel form of $(\mathcal{L}^{-1})^\dagger = \mathcal{L}^{-1}$.

18.3 Zero modes and the modified Green's function

Suppose the normalized zero modes are $e_1, \dots, e_k$. Division by their eigenvalues is impossible, but the inverse on their orthogonal complement is well defined. Introduce

$$P_0(x, s) = \sum_{j=1}^k e_j(x) e_j(s)^*, \quad (18.10)$$

$$G_\perp(x, s) = \sum_{\lambda_n \neq 0} \frac{e_n(x) e_n(s)^*}{\lambda_n}. \quad (18.11)$$

Then

$$\mathcal{L}_x G_\perp(x, s) = \frac{\delta(x - s)}{w(s)} - P_0(x, s). \quad (18.12)$$

For a compatible source $\langle e_j, f \rangle_w = 0$, define

$$u_\perp(x) = \int_a^b G_\perp(x, s) f(s) w(s) ds.$$

Acting with $\mathcal{L}$ gives

$$\mathcal{L}u_\perp = f - \sum_{j=1}^k e_j \langle e_j, f \rangle_w = f.$$

This proves the representation and the compatibility requirement together. The most general solution is

$$u = u_\perp + \sum_{j=1}^k c_j e_j. \quad (18.13)$$

The choice $u_\perp \perp \ker \mathcal{L}$ gives one distinguished solution.

Example 18.1 (The Neumann modified kernel). For (17.6), $w = 1$ and $P_0(x, s) = 1$. A symmetric kernel with zero mean in x is

$$G_\perp(x, s) = \frac{x^2 + s^2}{2} - \max(x, s) + \frac{1}{3}. \quad (18.14)$$

Away from $x = s$, $-\partial_x^2 G_\perp = -1$. At $x = s$, the derivative jumps by -1 , adding the required delta function. Moreover, $\partial_x G_\perp(0, s) = \partial_x G_\perp(1, s) = 0$ and $\int_0^1 G_\perp(x, s) dx = 0$. Thus $-\partial_x^2 G_\perp = \delta(x - s) - 1$, as required.

19 Constructing a Green's function from homogeneous solutions

19.1 Continuity and the derivative jump

Consider the Sturm–Liouville expression in the sign convention

$$\mathcal{L}u = \frac{1}{w(x)} [(p(x)u')' - q(x)u]. \quad (19.1)$$

For the weighted kernel of (18.8), multiplication by $w(x)$ gives

$$(pG_x)' - qG = \delta(x - s). \quad (19.2)$$

Away from $x = s$, the right side vanishes. Integrating across a short interval containing s yields

$$[p(x)G_x(x, s)]_{s-\epsilon}^{s+\epsilon} - \int_{s-\epsilon}^{s+\epsilon} q(x)G(x, s) dx = 1.$$

For a bounded kernel and regular coefficients, the integral tends to zero. Hence the two matching conditions are

$$G(s^+, s) = G(s^-, s), \quad (19.3)$$

$$p(s)[G_x(s^+, s) - G_x(s^-, s)] = 1. \quad (19.4)$$

The continuity condition can also be deduced distributionally: a jump in G would generate a δ' term in $(pG_x)'$, while the prescribed source contains only δ .

One can obtain the same jump without interchanging singular derivatives. Split the solution at its moving observation point:

$$u(x) = \int_a^x G_>(x, s)F(s) ds + \int_x^b G_<(x, s)F(s) ds, \quad F = wf,$$

where $G_>$ denotes the branch $x > s$. Leibniz's rule gives no endpoint term in u' because $G_> = G_<$ at $x = s$. Differentiating pu' produces the endpoint term

$$p(x)[\partial_x G_>(x, x) - \partial_x G_<(x, x)]F(x) = F(x),$$

while the remaining integrals contain the homogeneous equation. Thus $(pu')' - qu = F$. The jump is exactly the term that ordinary differentiation under an unsplit integral would miss.

19.2 The Wronskian formula

Let U solve the homogeneous equation and satisfy the left endpoint condition, and let V solve it and satisfy the right endpoint condition:

$$(pU')' - qU = 0, \quad (pV')' - qV = 0.$$

For separated homogeneous boundary conditions, set

$$G(x, s) = \begin{cases} A(s)U(x), & x < s, \\ B(s)V(x), & x > s. \end{cases} \quad (19.5)$$

The endpoint conditions are already satisfied. Continuity and the jump give

$$A(s)U(s) = B(s)V(s), \quad p(s)[B(s)V'(s) - A(s)U'(s)] = 1.$$

With

$$W(x) = U(x)V'(x) - U'(x)V(x), \quad (19.6)$$

the solution is

$$G(x, s) = \frac{U(x_<)V(x_>)}{p(s)W(s)}, \quad x_< = \min(x, s), \quad x_> = \max(x, s). \quad (19.7)$$

The denominator is constant. Indeed,

$$\begin{aligned} (pW)' &= U(pV')' - V(pU')' \\ &= UqV - VqU = 0. \end{aligned}$$

This is Abel's identity in self-adjoint form. Rescaling U or V rescales the numerator and denominator equally, so the kernel is independent of how these homogeneous solutions are normalized.

If $pW = 0$, the two solutions are linearly dependent. There is then a nonzero homogeneous solution satisfying both endpoint conditions: a zero mode. The failure of the formula is the same obstruction detected by the Fredholm alternative. For the opposite operator $w^{-1}[-(pu')' + qu]$, the jump and the kernel have the opposite sign.

19.3 Worked example: Dirichlet inverse of the second derivative

Solve

$$-u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0. \quad (19.8)$$

For $x \neq s$, the kernel is linear in x . The boundary conditions suggest

$$G(x, s) = \begin{cases} A(s)x, & x < s, \\ B(s)(x - 1), & x > s. \end{cases}$$

Continuity and the derivative jump for $-\partial_x^2$ give

$$A(s)s = B(s)(s - 1), \quad B(s) - A(s) = -1.$$

Solving yields $A(s) = 1 - s$ and $B(s) = -s$, hence

$$G(x, s) = \begin{cases} x(1 - s), & 0 \leq x \leq s, \\ s(1 - x), & s \leq x \leq 1. \end{cases} \quad (19.9)$$

For fixed s , this is a triangular profile, with peak $s(1 - s)$ at the source. It is continuous and symmetric under $x \leftrightarrow s$.

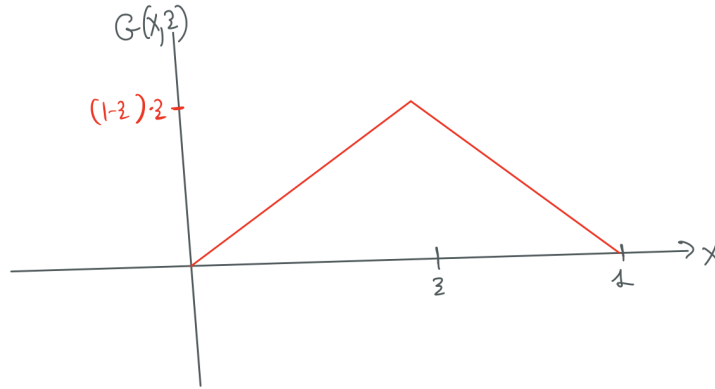


Figure 14: The Dirichlet Green's function of $-d^2/dx^2$ on the unit interval, with the source position marked in the original sketch.

To evaluate the response, split the integral according to whether the source point s lies to the left or right of the observation point x :

$$u(x) = (1 - x) \int_0^x s f(s) ds + x \int_x^1 (1 - s) f(s) ds. \quad (19.10)$$

Writing the two integrals as $A(x)$ and $B(x)$, differentiation gives

$$u' = -A + B, \quad u'' = -x f(x) - (1 - x) f(x) = -f(x).$$

The terms proportional to $x(1-x)f(x)$ cancel in the first derivative, and the endpoint values are zero. For the simple source $f = 1$, this gives $u = x(1-x)/2$.

The same kernel follows from the eigenfunctions $e_n = \sqrt{2} \sin(n\pi x)$ with eigenvalues $(n\pi)^2$:

$$G(x, s) = 2 \sum_{n=1}^{\infty} \frac{\sin(n\pi x) \sin(n\pi s)}{(n\pi)^2}. \quad (19.11)$$

Thus the piecewise construction and spectral construction are two representations of the same inverse.

19.4 Worked example: mixed boundary conditions

Consider the problem from the Green's-function handout:

$$u'' - \alpha^2 u = f(x), \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0, \quad \alpha > 0. \quad (19.12)$$

Choose

$$U(x) = \sinh(\alpha x), \quad V(x) = \cosh[\alpha(1-x)].$$

These satisfy the left and right boundary conditions respectively. Their Wronskian is

$$\begin{aligned} W &= -\alpha \sinh(\alpha x) \sinh[\alpha(1-x)] - \alpha \cosh(\alpha x) \cosh[\alpha(1-x)] \\ &= -\alpha \cosh \alpha. \end{aligned}$$

Consequently,

$$G(x, s) = -\frac{\sinh(\alpha x_{<}) \cosh[\alpha(1-x_{>})]}{\alpha \cosh \alpha}. \quad (19.13)$$

For $f(x) = x$, the integral solution is

$$\begin{aligned} u(x) = -\frac{1}{\alpha \cosh \alpha} \Big\{ &\cosh[\alpha(1-x)] \int_0^x s \sinh(\alpha s) \, ds \\ &+ \sinh(\alpha x) \int_x^1 s \cosh[\alpha(1-s)] \, ds \Big\}. \end{aligned} \quad (19.14)$$

The first integral is

$$\int_0^x s \sinh(\alpha s) \, ds = \frac{x \cosh(\alpha x)}{\alpha} - \frac{\sinh(\alpha x)}{\alpha^2}.$$

For the second, the substitution $y = 1 - s$ gives

$$\int_x^1 s \cosh[\alpha(1-s)] \, ds = \frac{x \sinh[\alpha(1-x)]}{\alpha} + \frac{\cosh[\alpha(1-x)] - 1}{\alpha^2}.$$

Substitute these expressions into (19.14) and use the hyperbolic addition formula. The result is

$$u(x) = \frac{\sinh(\alpha x)}{\alpha^3 \cosh \alpha} - \frac{x}{\alpha^2}. \quad (19.15)$$

Indeed,

$$u'(x) = \frac{\cosh(\alpha x)}{\alpha^2 \cosh \alpha} - \frac{1}{\alpha^2}, \quad u''(x) = \frac{\sinh(\alpha x)}{\alpha \cosh \alpha},$$

so $u'' - \alpha^2 u = x$, $u(0) = 0$, and $u'(1) = 0$. Expanding at small α also gives the finite limit

$$u(x) \longrightarrow \frac{x^3}{6} - \frac{x}{2},$$

which solves $u'' = x$ with the same mixed conditions.

20 Causal response of a driven oscillator

20.1 Response to an impulse and to a general force

A mass attached to a spring and a linear damper obeys

$$m\ddot{x} + b\dot{x} + kx = F(t), \quad 2\gamma = \frac{b}{m}, \quad \omega_0^2 = \frac{k}{m}. \quad (20.1)$$

Thus the normalized operator is $\mathcal{L}_t = \partial_t^2 + 2\gamma\partial_t + \omega_0^2$.

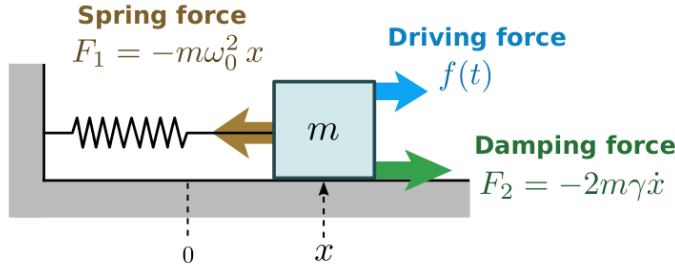


Figure 15: The spring, damping, and external driving forces in the oscillator model.

Define the retarded kernel by

$$\mathcal{L}_t G_R(t, t') = \delta(t - t'), \quad G_R(t, t') = 0 \quad (t < t'). \quad (20.2)$$

The condition $t < t'$ means that the system cannot respond before an impulse has been applied. Since the coefficients do not depend explicitly on time, $G_R(t, t') = G_R(t - t')$. For a system initially at rest before the force acts,

$$x(t) = \frac{1}{m} \int_{-\infty}^t G_R(t - t') F(t') dt'. \quad (20.3)$$

Applying $\mathcal{L}_t$ to the integral recovers $F(t)/m$. In particular, $F(t) = J\delta(t - t_0)$ produces $x(t) = JG_R(t - t_0)/m$. Here J is the impulse, with units of momentum; the resulting velocity change is J/m .

20.2 Matching at the impulse

Let $\tau = t - t'$. The kernel vanishes for $\tau < 0$, and obeys the homogeneous oscillator equation for $\tau > 0$. It is continuous at zero. Integrating its equation across zero yields

$$[G'_R]_{0-}^{0+} + 2\gamma[G_R]_{0-}^{0+} + \lim_{\epsilon \rightarrow 0} \omega_0^2 \int_{-\epsilon}^{\epsilon} G_R d\tau = 1.$$

Therefore

$$G_R(0^+) = 0, \quad G'_R(0^+) = 1. \quad (20.4)$$

The force impulse changes velocity, not position. These two conditions uniquely determine the homogeneous motion after the impulse, proving uniqueness of the causal kernel.

More explicitly, write $G_R(\tau) = \Theta(\tau)g(\tau)$. Distributional differentiation gives

$$G'_R = \Theta g' + g(0)\delta, \quad G''_R = \Theta g'' + g'(0)\delta + g(0)\delta'.$$

The absence of a δ' term requires $g(0) = 0$, and the coefficient of δ then fixes $g'(0) = 1$.

20.3 Fourier solution and contour choice

We use the temporal transform convention

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt, \quad f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega). \quad (20.5)$$

Integration by parts gives $\tilde{\dot{f}} = -i\omega \tilde{f}$ and $\tilde{\ddot{f}} = -\omega^2 \tilde{f}$, with vanishing endpoint terms or in the distributional interpretation. Since $\tilde{\delta} = 1$,

$$\tilde{G}_R(\omega) = \frac{1}{\omega_0^2 - \omega^2 - 2i\gamma\omega} = -\frac{1}{(\omega - \omega_+)(\omega - \omega_-)}, \quad (20.6)$$

where

$$\omega_{\pm} = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \quad (20.7)$$

For positive damping and $\omega_0 > 0$, both poles lie in the lower half-plane. Inverting gives

$$G_R(\tau) = - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{(\omega - \omega_+)(\omega - \omega_-)}.$$

If $\tau < 0$, the exponential decays on an upper semicircle. Closing upward encloses no poles, so $G_R = 0$. If $\tau > 0$, close downward. The contour is clockwise, so the real-axis integral is $-2\pi i$ times the sum of residues. The extra minus sign in the integrand gives

$$G_R(\tau) = i\Theta(\tau) \left[\frac{e^{-i\omega_+\tau}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_-\tau}}{\omega_- - \omega_+} \right]. \quad (20.8)$$

This calculation links the lower-half-plane pole locations to both temporal causality and decay after the impulse.

20.4 Underdamped, overdamped, and critically damped motion

When $0 \leq \gamma < \omega_0$, put $\Omega = \sqrt{\omega_0^2 - \gamma^2}$. Equation (20.8) becomes

$$G_R(\tau) = \Theta(\tau) e^{-\gamma\tau} \frac{\sin(\Omega\tau)}{\Omega}. \quad (20.9)$$

The oscillation frequency is Ω , and the envelope decays on the time scale $1/\gamma$.

When $\gamma > \omega_0$, let $\kappa = \sqrt{\gamma^2 - \omega_0^2}$. Then

$$G_R(\tau) = \Theta(\tau) e^{-\gamma\tau} \frac{\sinh(\kappa\tau)}{\kappa} = \frac{\Theta(\tau)}{2\kappa} [e^{-(\gamma-\kappa)\tau} - e^{-(\gamma+\kappa)\tau}]. \quad (20.10)$$

Since $0 < \kappa < \gamma$, both exponentials decay. The system returns to equilibrium without oscillating.

At critical damping, $\gamma = \omega_0$, the two poles coincide. Taking $\sin(\Omega\tau)/\Omega \rightarrow \tau$ as $\Omega \rightarrow 0$ gives

$$G_R(\tau) = \Theta(\tau) \tau e^{-\gamma\tau}. \quad (20.11)$$

This also follows directly from the repeated-root homogeneous solution $(A + B\tau)e^{-\gamma\tau}$ and the conditions $g(0) = 0$, $g'(0) = 1$. Thus the critical-damping exercise is the continuous limit of either neighboring regime.

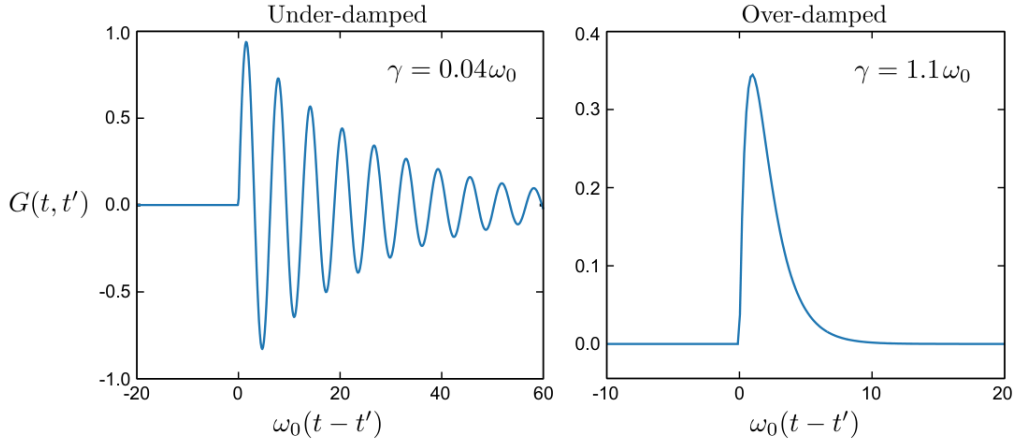


Figure 16: Original plots of the impulse response in underdamped and overdamped regimes. The response vanishes before the impulse and decays afterward.

In the undamped limit, the retarded prescription remains essential:

$$\tilde{G}_R(\omega) = \frac{1}{\omega_0^2 - (\omega + i0)^2}. \quad (20.12)$$

The poles approach the real axis from below. Omitting this limiting prescription would leave the inverse transform ambiguous.

20.5 Initial conditions, a step force, and periodic forcing

For initial data at a finite time t_0 , the complete solution is

$$x(t) = x_h(t) + \frac{1}{m} \int_{t_0}^t G_R(t-s) F(s) ds, \quad (20.13)$$

where x_h solves the homogeneous equation with the prescribed $x(t_0)$ and $\dot{x}(t_0)$. In the underdamped case,

$$x_h(t) = e^{-\gamma\tau} \left[x_0 \cos(\Omega\tau) + \frac{v_0 + \gamma x_0}{\Omega} \sin(\Omega\tau) \right], \quad \tau = t - t_0.$$

The source integral and its first derivative both vanish at $t = t_0$, so they do not change these initial conditions.

For a constant force switched on at zero, $F(t) = F_0\Theta(t)$, direct integration of (20.9) yields

$$x(t) = \frac{F_0}{m\omega_0^2} \left[1 - e^{-\gamma t} \left(\cos \Omega t + \frac{\gamma}{\Omega} \sin \Omega t \right) \right], \quad t \geq 0. \quad (20.14)$$

The displacement starts from zero with zero initial velocity and tends to F_0/k , the static equilibrium displacement.

For $F(t) = \text{Re}(F_\omega e^{-i\omega t})$, the steady response has the form

$$x(t) = \text{Re}[\chi(\omega) F_\omega e^{-i\omega t}], \quad \chi(\omega) = \frac{\tilde{G}_R(\omega)}{m}. \quad (20.15)$$

The distinction between $\tilde{G}_R$ and χ is the factor of m : the kernel G_R responds to force per unit mass, whereas χ responds to force itself.

20.6 Worked exercise: a random driving force

Suppose $\gamma > \omega_0 > 0$ and the force has zero mean and white-noise correlation

$$\langle F(t) \rangle = 0, \quad \langle F(t) F(t') \rangle = A \delta(t - t'). \quad (20.16)$$

Assume the force has acted from the distant past, so decaying initial transients have disappeared. Linearity gives $\langle x(t) \rangle = 0$. Using the convolution twice gives

$$\begin{aligned} \langle x(t_1) x(t_2) \rangle &= \frac{1}{m^2} \int_{-\infty}^{t_1} ds \int_{-\infty}^{t_2} ds' G_R(t_1 - s) G_R(t_2 - s') A \delta(s - s') \\ &= \frac{A}{m^2} \int_{-\infty}^{\min(t_1, t_2)} G_R(t_1 - s) G_R(t_2 - s) ds. \end{aligned} \quad (20.17)$$

Set $\tau = |t_1 - t_2|$ and $u = \min(t_1, t_2) - s$. Then

$$C(\tau) = \frac{A}{m^2} \int_0^\infty G_R(u + \tau) G_R(u) du.$$

Let $a = \gamma - \kappa$ and $b = \gamma + \kappa$, so $a + b = 2\gamma$, $ab = \omega_0^2$, and $b - a = 2\kappa$. Substitution of $G_R(u) = (e^{-au} - e^{-bu})/(b - a)$ gives four elementary integrals:

$$C(\tau) = \frac{A}{m^2(b-a)^2} \left[e^{-a\tau} \left(\frac{1}{2a} - \frac{1}{a+b} \right) + e^{-b\tau} \left(\frac{1}{2b} - \frac{1}{a+b} \right) \right]. \quad (20.18)$$

After simplifying,

$$C(\tau) = \frac{A}{4m^2\gamma\omega_0^2} e^{-\gamma\tau} \left[\cosh(\kappa\tau) + \frac{\gamma}{\kappa} \sinh(\kappa\tau) \right]. \quad (20.19)$$

Thus the stationary mean-square displacement is

$$\langle x^2 \rangle = C(0) = \frac{A}{4m^2\gamma\omega_0^2}. \quad (20.20)$$

The mean-square change over a time interval Δt follows from stationarity:

$$\begin{aligned} \langle [x(t + \Delta t) - x(t)]^2 \rangle &= 2[C(0) - C(|\Delta t|)] \\ &= \frac{A}{2m^2\gamma\omega_0^2} \left\{ 1 - e^{-\gamma|\Delta t|} \left[\cosh(\kappa|\Delta t|) + \frac{\gamma}{\kappa} \sinh(\kappa|\Delta t|) \right] \right\}. \end{aligned} \quad (20.21)$$

It vanishes at zero separation and approaches $2\langle x^2 \rangle$ for widely separated times. Expanding the bracket for small $|\Delta t|$ gives $\langle [x(t + \Delta t) - x(t)]^2 \rangle \sim A\Delta t^2/(4m^2\gamma)$, consistent with a finite velocity variance.

21 Causality, analyticity, and dispersion relations

21.1 Why a causal transform is analytic

For a linear time-translation-invariant response,

$$R(t) = \int_{-\infty}^{\infty} K(t - t') S(t') dt'. \quad (21.1)$$

Causality means $K(t) = 0$ for $t < 0$. Its frequency response is therefore

$$\tilde{K}(z) = \int_0^{\infty} K(t) e^{izt} dt, \quad z = \omega + i\eta. \quad (21.2)$$

When $\eta > 0$, the integrand contains the decaying factor $e^{-\eta t}$. For example, if $K \in L^1(0, \infty)$, then

$$|\tilde{K}(\omega + i\eta)| \leq \int_0^{\infty} |K(t)| e^{-\eta t} dt \leq \int_0^{\infty} |K(t)| dt.$$

On every closed region $\eta \geq \eta_0 > 0$, multiplication by powers of t is controlled by the exponential. Differentiation under the integral is then justified:

$$\frac{d^n \tilde{K}}{dz^n} = \int_0^{\infty} (it)^n K(t) e^{izt} dt. \quad (21.3)$$

Thus $\tilde{K}$ is analytic in the upper half-plane. More general stable kernels admit the same conclusion with suitable convergence or distributional assumptions. Causality supplies the one-sided integral; stability controls its large-time behavior.

For a real response kernel,

$$\tilde{K}(-\omega) = \tilde{K}(\omega)^*. \quad (21.4)$$

Its real part is even and its imaginary part is odd. These parity properties are useful both for deriving positive-frequency formulas and for checking signs in a calculation.

21.2 Boundary values of an analytic function

Let $F(z)$ be analytic in the upper half-plane, continuous on the real axis, and decreasing sufficiently fast at infinity that the large semicircle contributes zero. For a point z above the real axis, Cauchy's formula gives

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\omega')}{\omega' - z} d\omega'.$$

Approach the real axis from above, $z = \omega + i0$, and use

$$\frac{1}{x - i0} = \text{PV} \frac{1}{x} + i\pi\delta(x).$$

The boundary equation becomes

$$F(\omega) = \frac{1}{2\pi i} \text{PV} \int_{-\infty}^{\infty} \frac{F(\omega')}{\omega' - \omega} d\omega' + \frac{1}{2}F(\omega).$$

Moving the half-residue contribution to the left gives the dispersion relation

$$F(\omega) = \frac{1}{\pi i} \text{PV} \int_{-\infty}^{\infty} \frac{F(\omega')}{\omega' - \omega} d\omega'. \quad (21.5)$$

The principal value removes equal intervals on either side of $\omega' = \omega$; it is essential, not an ordinary improper integral.

Writing $F = F_R + iF_I$ and comparing real and imaginary parts yields the **Kramers–Kronig relations**:

$$F_R(\omega) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{F_I(\omega')}{\omega' - \omega} d\omega', \quad (21.6)$$

$$F_I(\omega) = -\frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{F_R(\omega')}{\omega' - \omega} d\omega'. \quad (21.7)$$

Each part determines the other by a Hilbert transform. Define

$$(\mathcal{H}f)(\omega) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(\omega')}{\omega - \omega'} d\omega'.$$

In this convention, $F_R = -\mathcal{H}F_I$ and $F_I = \mathcal{H}F_R$.

An immediate consequence is that an analytic, decaying function cannot have a nonzero, purely real boundary value on the whole real axis: $F_I = 0$ would imply $F_R = 0$. In a meromorphic example that is real on the real axis and decays at infinity, nonzero values therefore require a singularity away from that axis if the other contour assumptions hold. This is the contour observation behind the introduction of dispersion relations in the lecture.

If $F(\omega) \rightarrow F_\infty \neq 0$, apply these formulas to $F - F_\infty$, provided that the difference has the required decay. The constant corresponds to an instantaneous response proportional to $\delta(t)$ and must be separated before using an unsubtracted formula.

21.3 Positive-frequency form

For a real time-domain kernel, use the evenness of F_R and oddness of F_I to combine the integrals over positive and negative frequencies:

$$F_R(\omega) = \frac{2}{\pi} \text{PV} \int_0^\infty \frac{\omega' F_I(\omega')}{\omega'^2 - \omega^2} d\omega', \quad (21.8)$$

$$F_I(\omega) = -\frac{2\omega}{\pi} \text{PV} \int_0^\infty \frac{F_R(\omega')}{\omega'^2 - \omega^2} d\omega'. \quad (21.9)$$

For example, the real-part relation uses

$$\frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} = \frac{2\omega'}{\omega'^2 - \omega^2}.$$

At zero frequency this gives the useful sum rule

$$F_R(0) = \frac{2}{\pi} \int_0^\infty \frac{F_I(\omega')}{\omega'} d\omega', \quad (21.10)$$

when the integral converges.

21.4 The oscillator: reactive and dissipative response

The susceptibility in (20.15) has components

$$\chi_R(\omega) = \frac{\omega_0^2 - \omega^2}{m[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]}, \quad (21.11)$$

$$\chi_I(\omega) = \frac{2\gamma\omega}{m[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]}. \quad (21.12)$$

They have the correct parity and satisfy the Kramers–Kronig relations, because χ is analytic above the real axis and decays as $1/\omega^2$. The real part describes the component of displacement in phase with the force; the imaginary part describes the quadrature component.

The relation to dissipation follows directly. With complex amplitudes, $\dot{x}(t) = \text{Re}[-i\omega\chi F_\omega e^{-i\omega t}]$. The average over one period of $\text{Re}(Ae^{-i\omega t}) \text{Re}(Be^{-i\omega t})$ is $\frac{1}{2} \text{Re}(AB^*)$. Therefore

$$\overline{F\dot{x}} = \frac{\omega}{2} |F_\omega|^2 \chi_I(\omega) \geq 0 \quad (\omega > 0, \gamma > 0). \quad (21.13)$$

This is the power delivered to the damper in the steady state. The frequency dependence of this dissipative part and that of the reactive part cannot be chosen independently in a causal system.

21.5 Dielectric response and a necessary subtraction

For a homogeneous, local, isotropic dielectric, write the causal constitutive law as

$$\mathbf{D}(t) = \epsilon_0 \mathbf{E}(t) + \epsilon_0 \int_0^\infty \chi_e(\tau) \mathbf{E}(t - \tau) d\tau. \quad (21.14)$$

In frequency space,

$$\tilde{\mathbf{D}}(\omega) = \epsilon(\omega) \tilde{\mathbf{E}}(\omega), \quad \epsilon_r(\omega) = \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \tilde{\chi}_e(\omega). \quad (21.15)$$

If $\tilde{\chi}_e \rightarrow 0$ at high frequency, dispersion relations apply to $\epsilon_r - 1$:

$$\operatorname{Re} \epsilon_r(\omega) = 1 + \frac{1}{\pi} \operatorname{PV} \int_{-\infty}^{\infty} \frac{\operatorname{Im} \epsilon_r(\omega')}{\omega' - \omega} d\omega', \quad (21.16)$$

$$\operatorname{Im} \epsilon_r(\omega) = -\frac{1}{\pi} \operatorname{PV} \int_{-\infty}^{\infty} \frac{\operatorname{Re} \epsilon_r(\omega') - 1}{\omega' - \omega} d\omega'. \quad (21.17)$$

The real part controls dispersive phase propagation. The imaginary part controls loss from the coherent field, including absorption and, in an effective description, extinction by scattering. The relations connect these apparently different effects through causality. If the reference high-frequency permittivity is a different real constant over the model's range of validity, that constant replaces 1 in the subtraction.

Part V

Fourier Analysis

Fourier
analysis

22 The Fourier transform as a change of basis

22.1 Plane waves and the continuous spectrum

On the whole real line, the eigenfunctions of $-d^2/dx^2$ are plane waves e^{ikx} , with eigenvalue k^2 . The spectral label k is continuous because there are no finite-interval boundary conditions that quantize it. Plane waves are not square integrable on $\mathbb{R}$, so they are understood as generalized eigenfunctions. Their normalization is expressed by delta functions rather than Kronecker deltas:

$$\int_{-\infty}^{\infty} e^{i(k-k')x} dx = 2\pi\delta(k - k'). \quad (22.1)$$

A square-integrable function can be decomposed into this continuous basis:

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad (22.2)$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \widehat{f}(k) e^{ikx}. \quad (22.3)$$

For sufficiently regular, integrable functions these are ordinary integrals; for general L^2 functions the transform and inverse are defined by their L^2 limits. The completeness relation follows by substituting the forward transform into the inverse:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x - x'). \quad (22.4)$$

The sign in the spatial forward transform is negative. The temporal convention in (20.5) instead has a positive forward sign; this keeps a traveling wave proportional to $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$.

22.2 Generalization to several dimensions

For $\mathbf{r} = (x_1, \dots, x_d)$ and $\mathbf{k} = (k_1, \dots, k_d)$, define

$$\widehat{f}(\mathbf{k}) = \int_{\mathbb{R}^d} f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^d r, \quad (22.5)$$

$$f(\mathbf{r}) = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (22.6)$$

Here $d^d r = dx_1 \cdots dx_d$, and likewise for $d^d k$. Since

$$-\nabla^2 e^{i\mathbf{k}\cdot\mathbf{r}} = |\mathbf{k}|^2 e^{i\mathbf{k}\cdot\mathbf{r}},$$

the multidimensional Fourier transform diagonalizes the Laplacian. The identity kernel is

$$\delta^{(d)}(\mathbf{r} - \mathbf{r}') = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}. \quad (22.7)$$

22.3 The adjoint, normalization, and Parseval's identity

With the convention (22.5), denote the unnormalized forward operator by $\mathcal{F}_0$. For the usual L^2 inner products,

$$\begin{aligned}\langle g, \mathcal{F}_0 f \rangle &= \int d^d k g(\mathbf{k})^* \int d^d r e^{-i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}) \\ &= \int d^d r \left[\int e^{i\mathbf{k}\cdot\mathbf{r}} g(\mathbf{k}) d^d k \right]^* f(\mathbf{r}).\end{aligned}$$

Thus

$$(\mathcal{F}_0^\dagger g)(\mathbf{r}) = \int e^{i\mathbf{k}\cdot\mathbf{r}} g(\mathbf{k}) d^d k = (2\pi)^d (\mathcal{F}_0^{-1} g)(\mathbf{r}). \quad (22.8)$$

The unnormalized transform is not literally unitary with these measures. The symmetric normalization

$$\mathcal{U} = (2\pi)^{-d/2} \mathcal{F}_0 \quad (22.9)$$

satisfies $\mathcal{U}^\dagger = \mathcal{U}^{-1}$ and is unitary. Equivalently, in the original convention,

$$\int_{\mathbb{R}^d} f(\mathbf{r})^* h(\mathbf{r}) d^d r = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \widehat{f}(\mathbf{k})^* \widehat{h}(\mathbf{k}). \quad (22.10)$$

Taking $h = f$ gives Parseval's identity,

$$\int |f|^2 d^d r = \frac{1}{(2\pi)^d} \int |\widehat{f}|^2 d^d k. \quad (22.11)$$

This extends the transform from a dense set of regular functions to all of $L^2(\mathbb{R}^d)$ and shows that it preserves information.

Unitary does not mean self-adjoint. Applying $\mathcal{U}$ twice gives $\mathcal{U}^2 f(\mathbf{r}) = f(-\mathbf{r})$, because the intermediate Fourier integral produces $\delta^{(d)}(\mathbf{r} + \mathbf{r}')$. Consequently $\mathcal{U}^4 = I$, whereas in general $\mathcal{U}^2 \neq I$.

23 Fourier properties and elementary transform calculations

23.1 Translations, modulation, and scale changes

A change of variable $\mathbf{s} = \mathbf{r} - \mathbf{a}$ gives

$$\mathcal{F}_0[f(\mathbf{r} - \mathbf{a})](\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{a}} \widehat{f}(\mathbf{k}). \quad (23.1)$$

A displacement in real space becomes a phase in Fourier space. Conversely, combining the exponential factors gives

$$\mathcal{F}_0[e^{i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})](\mathbf{k}) = \widehat{f}(\mathbf{k} - \mathbf{q}). \quad (23.2)$$

For a nonzero real scale a , let $\mathbf{s} = a\mathbf{r}$. The Jacobian is $d^d r = |a|^{-d} d^d s$, so

$$\mathcal{F}_0[f(a\mathbf{r})](\mathbf{k}) = |a|^{-d} \widehat{f}(\mathbf{k}/a). \quad (23.3)$$

A narrow spatial profile therefore has a broad Fourier profile.

Complex conjugation gives

$$\widehat{f^*}(\mathbf{k}) = \widehat{f}(-\mathbf{k})^*.$$

For real f , $\widehat{f}(-\mathbf{k}) = \widehat{f}(\mathbf{k})^*$. A real even function has a real even transform; a real odd function has a purely imaginary odd transform.

23.2 Derivatives and multiplication by a coordinate

Integrating by parts in one coordinate,

$$\begin{aligned}\widehat{\partial_j f}(\mathbf{k}) &= \int e^{-i\mathbf{k}\cdot\mathbf{r}} \partial_j f \, d^d r \\ &= ik_j \int e^{-i\mathbf{k}\cdot\mathbf{r}} f \, d^d r = ik_j \widehat{f}(\mathbf{k}),\end{aligned}$$

provided the boundary term vanishes. Repeating the operation gives

$$\widehat{\partial_j^n f} = (ik_j)^n \widehat{f}, \quad \widehat{-\nabla^2 f} = k^2 \widehat{f}. \quad (23.4)$$

Differentiating the transform with respect to k_j instead gives

$$\partial_{k_j} \widehat{f} = \int (-ix_j) e^{-i\mathbf{k}\cdot\mathbf{r}} f \, d^d r,$$

so

$$\widehat{x_j f} = i \partial_{k_j} \widehat{f}. \quad (23.5)$$

These formulas explain why constant-coefficient differential equations become algebraic equations in Fourier space.

23.3 Convolution and products

Define the convolution by

$$(f * g)(\mathbf{r}) = \int_{\mathbb{R}^d} f(\mathbf{r} - \mathbf{s}) g(\mathbf{s}) \, d^d s. \quad (23.6)$$

Set $\mathbf{u} = \mathbf{r} - \mathbf{s}$ inside its transform:

$$\begin{aligned}\widehat{f * g}(\mathbf{k}) &= \int d^d s \int d^d u \, e^{-i\mathbf{k}\cdot(\mathbf{u}+\mathbf{s})} f(\mathbf{u}) g(\mathbf{s}) \\ &= \widehat{f}(\mathbf{k}) \widehat{g}(\mathbf{k}).\end{aligned}$$

Thus

$$\widehat{f * g} = \widehat{f} \widehat{g}, \quad \widehat{fg}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int \widehat{f}(\mathbf{q}) \widehat{g}(\mathbf{k} - \mathbf{q}) \, d^d q. \quad (23.7)$$

The second identity follows by inserting the inverse transforms of f and g and performing the spatial integral. It is the product–convolution duality. In a translation-invariant system, the Green’s representation is a convolution, so the response is simply a product in Fourier space.

23.4 Gaussian and exponential examples

For $a > 0$, let

$$I(k) = \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx.$$

Instead of shifting a complex contour, differentiate I with respect to k and integrate by parts:

$$\begin{aligned} I'(k) &= -i \int x e^{-ax^2} e^{-ikx} dx \\ &= \frac{i}{2a} \int (e^{-ax^2})' e^{-ikx} dx = -\frac{k}{2a} I(k). \end{aligned}$$

Since $I(0) = \sqrt{\pi/a}$,

$$\widehat{e^{-ax^2}}(k) = \sqrt{\frac{\pi}{a}} e^{-k^2/(4a)}. \quad (23.8)$$

In d dimensions the integral separates, giving

$$\widehat{e^{-ar^2}}(\mathbf{k}) = \left(\frac{\pi}{a}\right)^{d/2} e^{-k^2/(4a)}.$$

For $a > 0$, split the transform of $e^{-a|x|}$ at the origin:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx} dx &= \int_0^{\infty} e^{-(a+ik)x} dx + \int_0^{\infty} e^{-(a-ik)x} dx \\ &= \frac{1}{a+ik} + \frac{1}{a-ik} = \frac{2a}{a^2 + k^2}. \end{aligned}$$

This example also shows how exponential spatial decay produces simple poles in the analytically continued transform.

24 Rotational symmetry and radial transforms

24.1 Why a radial function has a radial transform

Suppose $f(O\mathbf{r}) = f(\mathbf{r})$ for every rotation O . Then f depends only on $r = |\mathbf{r}|$. We claim that its transform depends only on $k = |\mathbf{k}|$. To prove this, take $\mathbf{k}' = O\mathbf{k}$ and change variables $\mathbf{r} = O\mathbf{s}$:

$$\begin{aligned} \widehat{f}(O\mathbf{k}) &= \int e^{-i(O\mathbf{k}) \cdot \mathbf{r}} f(\mathbf{r}) d^d r \\ &= \int e^{-i\mathbf{k} \cdot \mathbf{s}} f(O\mathbf{s}) |\det O| d^d s \\ &= \int e^{-i\mathbf{k} \cdot \mathbf{s}} f(\mathbf{s}) d^d s = \widehat{f}(\mathbf{k}). \end{aligned}$$

We used $O^T O = I$, $|\det O| = 1$, and the rotational symmetry of f . The same argument works for invariance under a subgroup of rotations: the transform inherits that symmetry.

24.2 Three dimensions: reduction to a sine transform

Let $f(\mathbf{r}) = f(r)$ and write its transform as $g(k)$. Align the polar axis with the fixed vector $\mathbf{k}$, so $\mathbf{k} \cdot \mathbf{r} = kr \cos \theta$. In spherical coordinates,

$$\begin{aligned} g(k) &= \int_0^\infty r^2 f(r) dr \int_0^{2\pi} d\phi \int_0^\pi e^{-ikr \cos \theta} \sin \theta d\theta \\ &= 2\pi \int_0^\infty r^2 f(r) dr \int_{-1}^1 e^{-ikru} du. \end{aligned}$$

The inner integral is

$$\int_{-1}^1 e^{-ikru} du = \frac{e^{-ikr} - e^{ikr}}{-ikr} = \frac{2 \sin(kr)}{kr}.$$

Therefore

$$g(k) = 4\pi \int_0^\infty r^2 f(r) \frac{\sin(kr)}{kr} dr = \frac{4\pi}{k} \int_0^\infty r f(r) \sin(kr) dr. \quad (24.1)$$

The $k = 0$ value is taken by continuity and equals $4\pi \int_0^\infty r^2 f(r) dr$, when that integral exists. Apply the same angular integration to the inverse transform:

$$f(r) = \frac{1}{2\pi^2 r} \int_0^\infty k g(k) \sin(kr) dk. \quad (24.2)$$

The radial kernel is the spherical Bessel function $j_0(z) = \sin z/z$. The three-dimensional problem has become a pair of one-dimensional sine transforms of $rf(r)$ and $kg(k)$.

24.3 Coulomb and screened Coulomb kernels

For $f(r) = 1/r$, direct substitution gives an oscillatory integral that requires a limiting prescription. Introduce an exponential regulator, $f_\epsilon(r) = e^{-\epsilon r}/r$, with $\epsilon > 0$. Then

$$\begin{aligned} \int_0^\infty e^{-\epsilon r} \sin(kr) dr &= \text{Im} \int_0^\infty e^{-(\epsilon - ik)r} dr \\ &= \text{Im} \frac{1}{\epsilon - ik} = \frac{k}{k^2 + \epsilon^2}. \end{aligned}$$

Using (24.1),

$$\frac{\widehat{e^{-\epsilon r}}}{r}(\mathbf{k}) = \frac{4\pi}{k^2 + \epsilon^2}. \quad (24.3)$$

Taking $\epsilon \rightarrow 0^+$ in the distributional sense gives

$$\frac{\widehat{1}}{r}(\mathbf{k}) = \frac{4\pi}{k^2}. \quad (24.4)$$

The singularity at $k = 0$ is part of the distributional transform and reflects the long-range nature of $1/r$.

The same pair can be checked by applying a differential operator. Since multiplication by $k^2 + \epsilon^2$ is the transform of $-\nabla^2 + \epsilon^2$,

$$(-\nabla^2 + \epsilon^2) \frac{e^{-\epsilon r}}{4\pi r} = \delta^{(3)}(\mathbf{r}).$$

At $\epsilon = 0$ this reduces to $-\nabla^2(1/4\pi r) = \delta^{(3)}(\mathbf{r})$, the fundamental solution used in electrostatics.

24.4 Two dimensions: the order-zero Hankel transform

In the plane, choose the polar angle relative to $\mathbf{k}$ and use $d^2r = r dr d\phi$:

$$g(k) = \int_0^\infty r f(r) dr \int_0^{2\pi} e^{-ikr \cos \phi} d\phi. \quad (24.5)$$

The imaginary part of the angular integral vanishes. Indeed, the replacement $\phi \mapsto \phi + \pi$ sends $\cos \phi$ to $-\cos \phi$, so the sine terms cancel in pairs while the cosine terms add. Define

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-iz \cos \phi} d\phi = \frac{1}{2\pi} \int_0^{2\pi} \cos(z \cos \phi) d\phi. \quad (24.6)$$

Thus the radial transform pair in two dimensions is

$$g(k) = 2\pi \int_0^\infty r f(r) J_0(kr) dr, \quad (24.7)$$

$$f(r) = \frac{1}{2\pi} \int_0^\infty k g(k) J_0(kr) dk. \quad (24.8)$$

This is the order-zero Hankel, or Fourier–Bessel, transform. The radial measures $r dr$ and $k dk$ are essential.

25 Angular sectors and the Hankel transform

25.1 Plane-wave and cylindrical bases

The two-dimensional Laplacian can be written in Cartesian or polar form:

$$-\nabla^2 = -\partial_x^2 - \partial_y^2 = -\frac{1}{r} \partial_r (r \partial_r) - \frac{1}{r^2} \partial_\phi^2. \quad (25.1)$$

In Cartesian coordinates, separated generalized eigenfunctions are plane waves. In polar coordinates, seek $u(r, \phi) = R(r)\Phi(\phi)$ with $-\nabla^2 u = k^2 u$. Separation gives

$$\Phi'' + m^2 \Phi = 0, \quad r^2 R'' + r R' + (k^2 r^2 - m^2) R = 0.$$

Single-valuedness under $\phi \mapsto \phi + 2\pi$ requires $m \in \mathbb{Z}$, and the regular radial solution is $R(r) = J_m(kr)$. Thus the cylindrical basis is

$$J_m(kr) e^{im\phi}, \quad m \in \mathbb{Z}, \quad k \geq 0. \quad (25.2)$$

There is one discrete angular label and one continuous radial label.

For a fixed m , define the Hankel transform

$$(\mathcal{H}_m F)(k) = \int_0^\infty F(r) J_m(kr) r \, dr, \quad (25.3)$$

$$F(r) = \int_0^\infty (\mathcal{H}_m F)(k) J_m(kr) k \, dk. \quad (25.4)$$

Its distributional orthogonality and completeness relations are

$$\int_0^\infty J_m(kr) J_m(k'r) r \, dr = \frac{\delta(k - k')}{k}, \quad (25.5)$$

$$\int_0^\infty J_m(kr) J_m(kr') k \, dk = \frac{\delta(r - r')}{r}. \quad (25.6)$$

For example, inserting (25.3) into (25.4) gives

$$\int_0^\infty F(r') r' \, dr' \int_0^\infty J_m(kr) J_m(kr') k \, dk = \int_0^\infty F(r') \frac{\delta(r - r')}{r} r' \, dr' = F(r).$$

Orthogonality between different m values comes from the angular integral $\int_0^{2\pi} e^{i(m-n)\phi} d\phi = 2\pi\delta_{mn}$; it should not be inserted into a radial integral with arbitrary orders.

To connect this basis to the full Fourier transform, write $f(r, \phi) = \sum_m f_m(r) e^{im\phi}$ and let θ be the polar angle of $\mathbf{k}$. The angular identity

$$\int_0^{2\pi} e^{im\phi} e^{-ikr \cos(\phi-\theta)} d\phi = 2\pi(-i)^m e^{im\theta} J_m(kr)$$

gives

$$\hat{f}(k, \theta) = 2\pi \sum_{m \in \mathbb{Z}} (-i)^m e^{im\theta} \int_0^\infty f_m(r) J_m(kr) r \, dr. \quad (25.7)$$

The radial transform in each angular sector is therefore precisely $\mathcal{H}_m$, up to the known phase and normalization.

25.2 A Laplace–Bessel identity and its inverse

For $a > 0$ and $k \geq 0$, consider

$$I(a, k) = \int_0^\infty e^{-ar} J_0(kr) \, dr.$$

Use the angular representation of J_0 and interchange the absolutely convergent integrals:

$$I(a, k) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{a + ik \cos \phi}.$$

The imaginary part cancels. Taking the real part and using quadrant symmetry,

$$\begin{aligned} I(a, k) &= \frac{2a}{\pi} \int_0^{\pi/2} \frac{d\phi}{a^2 + k^2 \cos^2 \phi} \\ &= \frac{2a}{\pi} \int_0^\infty \frac{du}{a^2(1+u^2) + k^2} = \frac{1}{\sqrt{a^2 + k^2}}, \end{aligned}$$

where $u = \tan \phi$. Thus

$$\int_0^\infty e^{-ar} J_0(kr) dr = \frac{1}{\sqrt{a^2 + k^2}}. \quad (25.8)$$

This is the order-zero Hankel transform of e^{-ar}/r . Its inverse is

$$\int_0^\infty \frac{k J_0(kr)}{\sqrt{a^2 + k^2}} dk = \frac{e^{-ar}}{r}, \quad r > 0. \quad (25.9)$$

The inverse is understood as an oscillatory integral or a regulated limit. Differentiating (25.8) with respect to a produces the useful absolutely convergent identity

$$\int_0^\infty r e^{-ar} J_0(kr) dr = \frac{a}{(a^2 + k^2)^{3/2}}. \quad (25.10)$$

For example, the two-dimensional transform of e^{-ar} is therefore $2\pi a/(a^2 + k^2)^{3/2}$.

25.3 Oscillatory Bessel integrals and branch selection

The lecture also gives cosine and sine versions of (25.8). Introduce a convergence factor and continue a to $\epsilon - ib$, where $b > 0$ and $\epsilon > 0$:

$$\int_0^\infty e^{-\epsilon r} e^{ibr} J_0(kr) dr = \frac{1}{\sqrt{k^2 + (\epsilon - ib)^2}}. \quad (25.11)$$

The square root is fixed by continuation from $\text{Re } a > 0$. If $k > b$, the limiting denominator is the positive root $\sqrt{k^2 - b^2}$. If $0 \leq k < b$, its argument approaches the negative real axis from below, so the square root tends to $-i\sqrt{b^2 - k^2}$ and its reciprocal to $i/\sqrt{b^2 - k^2}$. Comparing real and imaginary parts gives

$$\int_0^\infty \cos(br) J_0(kr) dr = \begin{cases} 0, & 0 \leq k < b, \\ (k^2 - b^2)^{-1/2}, & k > b, \end{cases} \quad (25.12)$$

$$\int_0^\infty \sin(br) J_0(kr) dr = \begin{cases} (b^2 - k^2)^{-1/2}, & 0 \leq k < b, \\ 0, & k > b. \end{cases} \quad (25.13)$$

The formulas hold as Abel limits, with the factor $e^{-\epsilon r}$ removed only after integration. The threshold $k = b$ is singular. Hankel inversion then gives the corresponding pair of useful identities:

$$\int_b^\infty \frac{k J_0(kr)}{\sqrt{k^2 - b^2}} dk = \frac{\cos(br)}{r}, \quad (25.14)$$

$$\int_0^b \frac{k J_0(kr)}{\sqrt{b^2 - k^2}} dk = \frac{\sin(br)}{r}, \quad r > 0. \quad (25.15)$$

The square roots and integration intervals encode which side of the branch point contributes. They must be kept together.

26 Delta functions in curvilinear coordinates

A delta function is defined by its action under the physical volume measure:

$$\int \delta^{(d)}(\mathbf{r} - \mathbf{r}_0) f(\mathbf{r}) d^d r = f(\mathbf{r}_0). \quad (26.1)$$

If coordinates $q_1, \dots, q_d$ have volume element $d^d r = J(q) dq_1 \cdots dq_d$, the local formula is

$$\delta^{(d)}(\mathbf{r} - \mathbf{r}_0) = \frac{\prod_{j=1}^d \delta(q_j - q_{0j})}{J(q)}. \quad (26.2)$$

The denominator cancels the Jacobian in the integral. It may equivalently be evaluated at the source point wherever the coordinates are nonsingular. For angular variables, the angular delta is periodic.

In Cartesian coordinates,

$$\delta^{(d)}(\mathbf{r} - \mathbf{r}_0) = \prod_{j=1}^d \delta(x_j - x_{0j}). \quad (26.3)$$

In two-dimensional polar coordinates, away from $r_0 = 0$,

$$\delta^{(2)}(\mathbf{r} - \mathbf{r}_0) = \frac{1}{r} \delta(r - r_0) \delta_{2\pi}(\phi - \phi_0). \quad (26.4)$$

In three-dimensional cylindrical coordinates,

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}_0) = \frac{1}{\rho} \delta(\rho - \rho_0) \delta_{2\pi}(\phi - \phi_0) \delta(z - z_0). \quad (26.5)$$

In spherical coordinates, away from the coordinate singularities,

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}_0) = \frac{\delta(r - r_0) \delta(\theta - \theta_0) \delta_{2\pi}(\phi - \phi_0)}{r^2 \sin \theta}. \quad (26.6)$$

At an origin or pole, angular coordinates do not uniquely identify a point; the invariant integration identity should be used rather than substituting into a singular coordinate formula. For a source at the origin, a convenient radial convention uses the half-line delta $\delta_+(r)$ defined by $\int_0^\infty \delta_+(r) h(r) dr = h(0)$. Then

$$\delta^{(2)}(\mathbf{r}) = \frac{\delta_+(r)}{2\pi r}, \quad \delta^{(3)}(\mathbf{r}) = \frac{\delta_+(r)}{4\pi r^2},$$

with angular integration performed over the full circle or sphere.

The factors of r , $r^2 \sin \theta$, and ρ are the same factors that appear in radial inner products, Bessel completeness relations, and Green's function normalization. Choosing a basis, its integration measure, and its delta normalization consistently is what makes all of these representations agree.

Part VI

Special Functions

Special
functions

27 Legendre polynomials and associated Legendre functions

27.1 Why special functions appear

Separation of variables turns a partial differential equation into a family of ordinary differential equations. The geometry determines the equations: Cartesian separation usually gives exponential or trigonometric functions; cylindrical separation gives Bessel functions; spherical separation gives Legendre functions and spherical harmonics. These functions are useful because they solve the differential equation and satisfy the orthogonality relations needed to match arbitrary boundary data.

The Legendre equation is

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0, \quad -1 < x < 1. \quad (27.1)$$

Equivalently,

$$-\frac{d}{dx} [(1 - x^2)y'] = \ell(\ell + 1)y.$$

Thus it is a Sturm–Liouville eigenvalue problem with weight one. Its endpoints are singular because the coefficient of y' inside the derivative vanishes there. Requiring the solution to be regular at both endpoints selects the eigenvalues $\ell(\ell + 1)$ with $\ell = 0, 1, 2, \dots$

27.2 Power series, polynomial termination, and normalization

To see how the polynomials arise, temporarily write the eigenvalue as λ and substitute $y = \sum_{n=0}^{\infty} a_n x^n$ into $(1 - x^2)y'' - 2xy' + \lambda y = 0$. Comparing coefficients of x^n gives

$$(n + 2)(n + 1)a_{n+2} + [\lambda - n(n + 1)]a_n = 0,$$

so

$$a_{n+2} = \frac{n(n + 1) - \lambda}{(n + 2)(n + 1)} a_n. \quad (27.2)$$

The even and odd coefficients form separate sequences. If $\lambda = \ell(\ell + 1)$, the sequence of the same parity as ℓ terminates at degree ℓ . The corresponding polynomial is normalized by $P_\ell(1) = 1$. The independent solution is singular at one or both endpoints and is excluded when a smooth angular function is required.

A compact expression for the regular polynomial is Rodrigues' formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell. \quad (27.3)$$

The first few polynomials are

$$\begin{aligned} P_0(x) &= 1, & P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), & P_3(x) &= \frac{1}{2}(5x^3 - 3x), \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned}$$

They have parity $P_\ell(-x) = (-1)^\ell P_\ell(x)$ and endpoint values $P_\ell(-1) = (-1)^\ell$.

To verify the differential equation directly, put $v = (x^2 - 1)^\ell$. Then $(x^2 - 1)v' = 2\ell xv$. Differentiate this identity $\ell + 1$ times:

$$(x^2 - 1)v^{(\ell+2)} + 2xv^{(\ell+1)} - \ell(\ell + 1)v^{(\ell)} = 0.$$

Dividing by $2^\ell \ell!$ gives (27.1). At $x = 1$, the only nonzero contribution to the ℓ th derivative of $(x - 1)^\ell(x + 1)^\ell$ is $\ell!2^\ell$, proving $P_\ell(1) = 1$.

27.3 Orthogonality and the normalization integral

For degrees ℓ and k , multiply their Sturm–Liouville equations by P_k and P_ℓ , subtract, and integrate. This gives

$$[\ell(\ell + 1) - k(k + 1)] \int_{-1}^1 P_\ell P_k \, dx = [(1 - x^2)(P_\ell P'_k - P_k P'_\ell)]_{-1}^1 = 0.$$

The boundary term vanishes because both functions are polynomials. Therefore polynomials of different degrees are orthogonal.

The norm can be evaluated without assuming its value. The leading coefficient of P_ℓ is

$$A_\ell = \frac{(2\ell)!}{2^\ell (\ell!)^2}.$$

Using Rodrigues' formula and integrating by parts ℓ times,

$$\begin{aligned} \int_{-1}^1 P_\ell(x)^2 \, dx &= \frac{(-1)^\ell}{2^\ell \ell!} \int_{-1}^1 P_\ell^{(\ell)}(x)(x^2 - 1)^\ell \, dx \\ &= \frac{A_\ell}{2^\ell} \int_{-1}^1 (1 - x^2)^\ell \, dx. \end{aligned}$$

All boundary terms vanish because $(x^2 - 1)^\ell$ and its first $\ell - 1$ derivatives vanish at both endpoints. The remaining integral is

$$\int_{-1}^1 (1 - x^2)^\ell \, dx = \frac{2^{2\ell+1}(\ell!)^2}{(2\ell + 1)!},$$

which follows, for example, from $x = 2t - 1$ and the beta integral. Consequently,

$$\int_{-1}^1 P_\ell(x)P_k(x) \, dx = \frac{2}{2\ell + 1} \delta_{\ell k}. \quad (27.4)$$

The normalized functions are $\sqrt{(2\ell + 1)/2} P_\ell$.

For an integrable function with a convergent Legendre expansion,

$$f(x) = \sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}(x), \quad c_{\ell} = \frac{2\ell+1}{2} \int_{-1}^1 f(x) P_{\ell}(x) dx.$$

For $f \in L^2[-1, 1]$, convergence is understood in the mean-square sense. Pointwise convergence requires additional regularity.

27.4 Generating function and recurrence relations

The generating function is

$$G(x, t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{\ell=0}^{\infty} P_{\ell}(x) t^{\ell}, \quad |t| < 1, \quad -1 \leq x \leq 1. \quad (27.5)$$

One way to identify its coefficients is to differentiate:

$$(1 - 2xt + t^2)G_t = (x - t)G.$$

Equating coefficients of t^{ℓ} yields

$$(\ell + 1)P_{\ell+1}(x) = (2\ell + 1)xP_{\ell}(x) - \ell P_{\ell-1}(x). \quad (27.6)$$

Together with the first coefficients $P_0 = 1$ and $P_1 = x$, this recurrence constructs the same normalized polynomials as Rodrigues' formula. Further differentiation gives the useful identities

$$(1 - x^2)P'_{\ell}(x) = \ell[P_{\ell-1}(x) - xP_{\ell}(x)], \quad (27.7)$$

$$(2\ell + 1)P_{\ell}(x) = P'_{\ell+1}(x) - P'_{\ell-1}(x). \quad (27.8)$$

The generating function will also turn the inverse-distance kernel into a multipole series.

27.5 Associated Legendre functions

Spherical separation with nonzero azimuthal dependence produces

$$\frac{d}{dx} [(1 - x^2)y'] + \left[\ell(\ell + 1) - \frac{m^2}{1 - x^2} \right] y = 0. \quad (27.9)$$

For integers $0 \leq m \leq \ell$, use the Condon–Shortley convention

$$P_{\ell}^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m P_{\ell}}{dx^m}. \quad (27.10)$$

The factor $(-1)^m$ is already included here. Differentiating the Legendre equation m times and multiplying by $(1 - x^2)^{m/2}$ verifies (27.9). Equivalently,

$$P_{\ell}^m(x) = \frac{(-1)^m}{2^{\ell} \ell!} (1 - x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^{\ell}.$$

For negative orders, define

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(x), \quad 0 \leq m \leq \ell. \quad (27.11)$$

The differential equation depends on m^2 , so these are proportional solutions of the same equation, rather than an additional independent regular solution. Regularity at both poles requires $|m| \leq \ell$.

For fixed m , the associated functions are orthogonal in ℓ :

$$\int_{-1}^1 P_\ell^m(x) P_k^m(x) dx = \frac{2}{2\ell + 1} \frac{(\ell + m)!}{(\ell - m)!} \delta_{\ell k}. \quad (27.12)$$

Orthogonality follows by the same subtraction of differential equations as before. To obtain the norm for $m \geq 0$, write the integrand using (27.10) and integrate by parts m times. The identity

$$\frac{d^m}{dx^m} \left[(1 - x^2)^m \frac{d^m P_\ell}{dx^m} \right] = (-1)^m \frac{(\ell + m)!}{(\ell - m)!} P_\ell$$

reduces the integral to the ordinary Legendre norm. This identity follows by repeated use of the differentiated Legendre equation. The factors $(1 - x^2)^m$ remove all boundary terms. The negative-order formula then extends the result to all allowed m .

28 Spherical separation and spherical harmonics

28.1 Separating Laplace's equation

In spherical coordinates,

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \quad (28.1)$$

Seek a nonzero separated solution of $\nabla^2 T = 0$ in the form $T(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$. Division by $R\Theta\Phi$ and multiplication by r^2 give

$$\frac{1}{R} \frac{d}{dr} (r^2 R') + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} (\sin \theta \Theta') + \frac{\Phi''}{\Phi \sin^2 \theta} = 0.$$

After multiplying by $\sin^2 \theta$, the ϕ -dependent term is separated from all the others. Thus $\Phi''/\Phi = -m^2$, and

$$\Phi'' + m^2 \Phi = 0.$$

A single-valued scalar field obeys $\Phi(\phi + 2\pi) = \Phi(\phi)$. Therefore m is an integer and the azimuthal solutions can be written $e^{im\phi}$. For $m = 0$, the ODE also has a solution proportional to ϕ , but periodicity excludes it. Only the constant solution remains.

Separating the remaining radial and angular dependence gives

$$\frac{d}{dr}(r^2 R') - \ell(\ell + 1)R = 0, \quad (28.2)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta}(\sin \theta \Theta') + \left[\ell(\ell + 1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0. \quad (28.3)$$

The radial equation is an Euler equation. Substitution of $R = r^s$ gives

$$s(s - 1) + 2s - \ell(\ell + 1) = 0, \quad (s - \ell)(s + \ell + 1) = 0,$$

so its two independent solutions are r^ℓ and $r^{-\ell-1}$. For the angular equation, setting $x = \cos \theta$ gives (27.9). Smoothness at the poles selects $\ell = 0, 1, 2, \dots$ and $m = -\ell, \dots, \ell$.

28.2 Normalization, phases, and explicit low-order functions

Define

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_\ell^m(\cos \theta) e^{im\phi}. \quad (28.4)$$

The Condon–Shortley sign is contained in P_ℓ^m . The conjugation rule is

$$Y_{\ell, -m} = (-1)^m Y_{\ell m}^*. \quad (28.5)$$

For $m = 0$,

$$Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} P_\ell(\cos \theta).$$

The functions with $\ell \leq 2$ and $m \geq 0$ are

$$\begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}}, \\ Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos \theta, & Y_{11} &= -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}, \\ Y_{20} &= \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1), \\ Y_{21} &= -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}, & Y_{22} &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\phi}. \end{aligned}$$

Negative orders follow from (28.5).

With $d\Omega = \sin \theta d\theta d\phi$, orthonormality reads

$$\int_{S^2} Y_{\ell m}^*(\Omega) Y_{\ell' m'}(\Omega) d\Omega = \delta_{\ell\ell'} \delta_{mm'}. \quad (28.6)$$

Indeed, the ϕ integral gives $2\pi \delta_{mm'}$, and the change of variables $x = \cos \theta$ reduces the remaining integral to (27.12). The prefactors in (28.4) cancel that norm exactly.

28.3 Angular differential operators and their eigenspaces

It is useful to separate the angular part of the Laplacian explicitly:

$$\nabla^2 = \frac{1}{r^2} \partial_r (r^2 \partial_r) - \frac{\Lambda^2}{r^2},$$

where

$$\Lambda^2 = -\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) - \frac{1}{\sin^2 \theta} \partial_\phi^2. \quad (28.7)$$

Then

$$\Lambda^2 Y_{\ell m} = \ell(\ell + 1) Y_{\ell m}, \quad -i \partial_\phi Y_{\ell m} = m Y_{\ell m}. \quad (28.8)$$

In quantum mechanics, $L^2 = \hbar^2 \Lambda^2$ and $L_z = -i \hbar \partial_\phi$. The angular eigenvalue $\ell(\ell + 1)$ has multiplicity $2\ell + 1$.

For smooth functions on the sphere, integration by parts gives

$$\int_{S^2} f^* \Lambda^2 g \, d\Omega = \int_{S^2} \left[(\partial_\theta f)^* \partial_\theta g + \frac{(\partial_\phi f)^* \partial_\phi g}{\sin^2 \theta} \right] d\Omega.$$

There is no boundary on the sphere; the apparent coordinate boundaries at the poles are handled by smoothness. This identity shows that Λ^2 is symmetric and nonnegative on smooth angular functions.

28.4 Laplace series and projection coefficients

The spherical harmonics are a complete orthonormal basis of $L^2(S^2)$. Thus

$$f(\Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\Omega), \quad a_{\ell m} = \int_{S^2} Y_{\ell m}^*(\Omega) f(\Omega) \, d\Omega. \quad (28.9)$$

The coefficient formula follows by multiplying the series by $Y_{\ell' m'}^*$ and integrating: all terms except $(\ell, m) = (\ell', m')$ vanish. The corresponding Parseval identity is

$$\int_{S^2} |f|^2 \, d\Omega = \sum_{\ell, m} |a_{\ell m}|^2.$$

Completeness can also be expressed as a distributional identity,

$$\sum_{\ell, m} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega') = \delta_{S^2}(\Omega, \Omega'),$$

where integration of the right-hand side against $f(\Omega') d\Omega'$ returns $f(\Omega)$.

If f is independent of ϕ , its projection onto every $m \neq 0$ mode vanishes because $\int_0^{2\pi} e^{-im\phi} d\phi = 0$. Writing $x = \cos \theta$,

$$a_{\ell 0} = \sqrt{\pi(2\ell + 1)} \int_{-1}^1 f(\arccos x) P_\ell(x) \, dx,$$

$$f(\theta) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cos \theta), \quad c_\ell = \frac{2\ell + 1}{2} \int_{-1}^1 f(\arccos x) P_\ell(x) \, dx.$$

Thus axial symmetry reduces a spherical-harmonic expansion to a Legendre series.

Proposition 28.1. *If a smooth angular function satisfies $\Lambda^2 f = \ell(\ell + 1)f$, then*

$$f(\Omega) = \sum_{m=-\ell}^{\ell} b_m Y_{\ell m}(\Omega).$$

Proof. Expand f in the complete basis. Applying Λ^2 and subtracting $\ell(\ell + 1)f$ gives

$$\sum_{k=0}^{\infty} \sum_{m=-k}^k [k(k+1) - \ell(\ell+1)] a_{km} Y_{km} = 0.$$

Projecting onto Y_{km} shows that every a_{km} with $k \neq \ell$ is zero. Only the $2\ell + 1$ modes of degree ℓ remain. $\square$

The general separated harmonic expansion in a spherical shell is

$$T(r, \Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1}) Y_{\ell m}(\Omega). \quad (28.10)$$

Regularity at $r = 0$ removes the $B_{\ell m}$ terms. Decay at infinity removes the $A_{\ell m}$ terms in an exterior problem.

Example 28.2 (Matching data on a sphere). Suppose $T(a, \Omega) = f(\Omega)$ and T is harmonic and regular for $r < a$. Expand f with coefficients $a_{\ell m}$ from (28.9). Matching each independent angular mode gives

$$T(r, \Omega) = \sum_{\ell, m} a_{\ell m} \left(\frac{r}{a}\right)^{\ell} Y_{\ell m}(\Omega).$$

For an exterior solution tending to zero,

$$T(r, \Omega) = \sum_{\ell, m} a_{\ell m} \left(\frac{a}{r}\right)^{\ell+1} Y_{\ell m}(\Omega).$$

For example, boundary data $f(\theta) = V \cos \theta$ contain only $\ell = 1, m = 0$; the interior solution is $Vr \cos \theta / a$ and the exterior solution is $Va^2 \cos \theta / r^2$.

29 The spherical-harmonic addition theorem

29.1 Geometry and the statement

Let $\Omega = (\theta, \phi)$ and $\Omega' = (\theta', \phi')$ describe two unit vectors. Their angle γ obeys

$$\cos \gamma = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'). \quad (29.1)$$

The addition theorem states

$$P_{\ell}(\cos \gamma) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega'). \quad (29.2)$$

For $\ell = 0$ it says $1 = 4\pi |Y_{00}|^2$. For $\ell = 1$, the $m = 0$ term is $\cos \theta \cos \theta'$, while the $m = \pm 1$ terms combine into $\sin \theta \sin \theta' \cos(\phi - \phi')$. Thus it reproduces the ordinary dot-product formula (29.1).

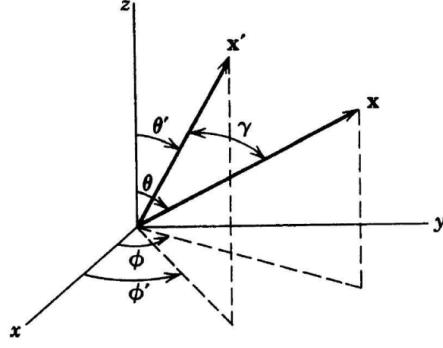


Figure 17: The two directions and their relative angle γ .

29.2 First proof: rotate the polar axis and compute the coefficients

Fix Ω' and choose a new coordinate system whose north pole lies along that direction. In the new coordinates (γ, β) ,

$$P_\ell(\cos \gamma) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\gamma, \beta).$$

Rotations leave the three-dimensional Laplacian and the radial distance unchanged. Consequently they leave the angular operator Λ^2 unchanged. Therefore $P_\ell(\cos \gamma)$, viewed as a function of the original variables (θ, ϕ) , belongs to the degree- ℓ eigenspace. Proposition 28.1 gives

$$P_\ell(\cos \gamma) = \sum_{m=-\ell}^{\ell} b_m(\Omega') Y_{\ell m}(\Omega),$$

with

$$b_m(\Omega') = \int_{S^2} P_\ell(\cos \gamma) Y_{\ell m}^*(\Omega) d\Omega. \quad (29.3)$$

This integral is awkward in the original coordinates, but its value can be obtained from the same fixed-eigenspace argument.

Expand the original function $Y_{\ell m}^*(\Omega)$ in the rotated basis:

$$Y_{\ell m}^*(\Omega) = \sum_{n=-\ell}^{\ell} B_{mn} Y_{\ell n}(\gamma, \beta).$$

The coefficient of the rotated $n = 0$ mode is

$$\begin{aligned} B_{m0} &= \int_{S^2} Y_{\ell m}^*(\Omega) Y_{\ell 0}(\gamma, \beta) d\Omega_\gamma \\ &= \sqrt{\frac{2\ell+1}{4\pi}} \int_{S^2} Y_{\ell m}^*(\Omega) P_\ell(\cos \gamma) d\Omega \\ &= \sqrt{\frac{2\ell+1}{4\pi}} b_m(\Omega'). \end{aligned}$$

Here $d\Omega_\gamma = d\Omega$ because rotations preserve spherical area.

Now evaluate the rotated expansion at its north pole, $\gamma = 0$. The factor $(1 - x^2)^{|n|/2}$ in the associated Legendre functions gives

$$Y_{\ell n}(0, \beta) = \delta_{n0} \sqrt{\frac{2\ell + 1}{4\pi}}.$$

Only $n = 0$ survives. At this point the original direction is precisely Ω' , so

$$Y_{\ell m}^*(\Omega') = B_{m0} \sqrt{\frac{2\ell + 1}{4\pi}} = \frac{2\ell + 1}{4\pi} b_m(\Omega').$$

Substitute $b_m = 4\pi Y_{\ell m}^*(\Omega')/(2\ell + 1)$ into the original expansion to obtain (29.2).

29.3 Second proof: orthogonality and equality in Cauchy's inequality

The kernel $P_\ell(\cos \gamma)$ is a degree- ℓ eigenfunction in either angular argument. Hence it admits a double expansion

$$P_\ell(\cos \gamma) = \sum_{m,n=-\ell}^{\ell} b_{mn} Y_{\ell m}(\Omega) Y_{\ell n}(\Omega').$$

Its dependence on the azimuthal angles is only through $\phi - \phi'$. Under $(\phi, \phi') \mapsto (\phi + \alpha, \phi' + \alpha)$ the kernel does not change. A product in the double sum acquires $e^{i(m+n)\alpha}$, so only terms with $n = -m$ survive. Using $Y_{\ell, -m} = (-1)^m Y_{\ell m}^*$, write

$$P_\ell(\cos \gamma) = \sum_{m=-\ell}^{\ell} c_m Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega'). \quad (29.4)$$

Reality of the kernel and symmetry under exchange of its two arguments imply $c_m \in \mathbb{R}$ and $c_m = c_{-m}$.

Set $\Omega' = \Omega$. Since $P_\ell(1) = 1$,

$$\sum_{m=-\ell}^{\ell} c_m |Y_{\ell m}(\Omega)|^2 = 1.$$

Integrating and using normalization yields

$$\sum_{m=-\ell}^{\ell} c_m = 4\pi. \quad (29.5)$$

Next take the absolute square of (29.4) and integrate over Ω . On the left, rotate the polar axis to Ω' and use Legendre orthogonality:

$$\int_{S^2} P_\ell(\cos \gamma)^2 d\Omega = 2\pi \int_{-1}^1 P_\ell(x)^2 dx = \frac{4\pi}{2\ell + 1}.$$

On the right, orthogonality removes all cross terms. Thus

$$\sum_{m=-\ell}^{\ell} c_m^2 |Y_{\ell m}(\Omega')|^2 = \frac{4\pi}{2\ell + 1}.$$

Integrating once more, now over Ω' , gives

$$\sum_{m=-\ell}^{\ell} c_m^2 = \frac{(4\pi)^2}{2\ell + 1}. \quad (29.6)$$

For n real numbers a_j , Cauchy's inequality says

$$\left(\sum_{j=1}^n a_j \right)^2 \leq n \sum_{j=1}^n a_j^2,$$

with equality exactly when all a_j are equal. Equations (29.5)–(29.6) saturate this inequality for $n = 2\ell + 1$. Therefore all c_m coincide and $c_m = 4\pi/(2\ell + 1)$, proving the theorem.

Proof of the inequality and its equality condition. Let $\bar{a} = n^{-1} \sum_j a_j$. Then

$$0 \leq \sum_{j=1}^n (a_j - \bar{a})^2 = \sum_j a_j^2 - \frac{1}{n} \left(\sum_j a_j \right)^2.$$

Equality holds exactly when every $a_j = \bar{a}$. Equivalently, the nonnegative quadratic

$$F(s) = \sum_j (a_j s - 1)^2 = s^2 \sum_j a_j^2 - 2s \sum_j a_j + n$$

has a nonpositive discriminant whenever $\sum a_j^2 > 0$. This gives the same inequality. If every $a_j = 0$, the inequality and its equality condition hold directly. The general Cauchy–Schwarz inequality follows by applying the same argument to $\sum_j (sa_j - b_j)^2$:

$$\left| \sum_j a_j b_j \right|^2 \leq \left(\sum_j a_j^2 \right) \left(\sum_j b_j^2 \right).$$

For nonzero real vectors, equality means that they are proportional. □

29.4 A rotation-invariant kernel interpretation

There is a useful operator interpretation of the same result. Define

$$K_{\ell}(\Omega, \Omega') = \sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega').$$

A rotation sends an orthonormal basis of the degree- ℓ eigenspace to another orthonormal basis of that same space. Thus the transformed basis is related by a unitary matrix D : $Y'_{\ell m} = \sum_n D_{mn} Y_{\ell n}$. In the sum defining K_ℓ ,

$$\sum_m D_{mn} D_{mp}^* = \delta_{np},$$

so the kernel is unchanged. Rotate Ω' to the north pole. Only its $m = 0$ component survives, and

$$K_\ell(\Omega, \Omega') = \frac{2\ell + 1}{4\pi} P_\ell(\cos \gamma).$$

This is the addition theorem again, now seen as the basis independence of an orthogonal projection.

At coincident directions it gives

$$\sum_{m=-\ell}^{\ell} |Y_{\ell m}(\Omega)|^2 = \frac{2\ell + 1}{4\pi}.$$

Moreover, for any function f_ℓ in this eigenspace,

$$\int_{S^2} K_\ell(\Omega, \Omega') f_\ell(\Omega') d\Omega' = f_\ell(\Omega).$$

30 Inverse-distance expansions and angular integrals

For two positions $\mathbf{r}$ and $\mathbf{r}'$, the law of cosines gives

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \gamma}}.$$

If $r' < r$, factor out r and apply the Legendre generating function with $t = r'/r$:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos \gamma).$$

If $r' > r$, interchange the two radii. With $r_< = \min(r, r')$ and $r_> = \max(r, r')$, both cases become

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos \gamma), \quad r_< < r_>. \quad (30.1)$$

The addition theorem then gives the fully separated kernel

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{\ell=0}^{\infty} \frac{1}{2\ell + 1} \frac{r_<^\ell}{r_>^{\ell+1}} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega'). \quad (30.2)$$

For unequal radii, the ratio $r_{<}/r_{>}$ is below one and the series is well behaved. At equal radii, the generating-function limit must be handled separately. For $0 < \gamma < \pi$ it gives the limiting value $[2r \sin(\gamma/2)]^{-1}$; at $\gamma = \pi$ the ordinary series alternates without converging, although its Abel limit exists. At $\gamma = 0$ the kernel itself is singular. Taking the limit from $r_{<}/r_{>} < 1$ avoids assuming uniform convergence on the coincident-radius sphere.

Example 30.1 (Angular average of the Coulomb kernel). Evaluate

$$I(r, r') = \int_{S^2} \frac{d\Omega}{|\mathbf{r} - \mathbf{r}'|}.$$

First choose the polar axis along $\mathbf{r}'$. Then $\gamma = \theta$ and

$$I = 2\pi \int_{-1}^1 \frac{d\mu}{\sqrt{r^2 + r'^2 - 2rr'\mu}}.$$

Put $z = r^2 + r'^2 - 2rr'\mu$. For nonzero r, r' ,

$$\begin{aligned} I &= \frac{\pi}{rr'} \int_{(r-r')^2}^{(r+r')^2} z^{-1/2} dz \\ &= \frac{2\pi}{rr'} (r + r' - |r - r'|) = \frac{4\pi}{r_{>}}. \end{aligned}$$

The cases with one radius zero follow by continuity.

A second method uses (30.2). Since $1 = \sqrt{4\pi} Y_{00}$,

$$\int_{S^2} Y_{\ell m}(\Omega) d\Omega = \sqrt{4\pi} \delta_{\ell 0} \delta_{m 0}.$$

Only the monopole term survives, immediately giving $I = 4\pi/r_{>}$. This explains why the angular average does not require the higher multipoles. The result also holds for equal positive radii because the point singularity is integrable over solid angle.

31 Bessel functions and cylindrical eigenfunctions

31.1 The Bessel equation from cylindrical separation

For a two-dimensional Helmholtz problem,

$$(\nabla^2 + k^2)u = 0, \quad \nabla^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2.$$

Writing $u = R(r)e^{im\phi}$ gives

$$r^2 R'' + rR' + (k^2 r^2 - m^2)R = 0.$$

With $z = kr$, this becomes Bessel's equation

$$z^2 y'' + zy' + (z^2 - \nu^2)y = 0, \tag{31.1}$$

where $\nu = |m|$ for a single-valued scalar on a full circle.

31.2 Frobenius series and independent solutions

Seek $y = \sum_{j=0}^{\infty} a_j z^{j+s}$ with $a_0 \neq 0$. The lowest-order coefficient gives $s^2 - \nu^2 = 0$, so $s = \pm\nu$. The remaining recurrence is

$$[(j+s)^2 - \nu^2]a_j + a_{j-2} = 0, \quad a_{-1} = a_{-2} = 0.$$

For the $s = \nu$ branch, only even powers beyond z^ν occur, and

$$a_{2j} = -\frac{a_{2j-2}}{4j(j+\nu)}.$$

Choose the normalization $a_0 = 2^{-\nu}/\Gamma(\nu+1)$. Iteration yields

$$J_\nu(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j+\nu+1)} \left(\frac{z}{2}\right)^{2j+\nu}. \quad (31.2)$$

For noninteger ν , J_ν and $J_{-\nu}$ are independent. A convenient second solution is

$$Y_\nu(z) = \frac{\cos(\pi\nu)J_\nu(z) - J_{-\nu}(z)}{\sin(\pi\nu)}.$$

For an integer n , $J_{-n} = (-1)^n J_n$, so define Y_n by taking the limit $\nu \rightarrow n$. The general solution for positive z is $AJ_\nu(z) + BY_\nu(z)$.

For $n \geq 0$, $J_n(z) \sim (z/2)^n/n!$ near zero. The independent Y_n solution is singular: Y_0 has a logarithmic singularity and Y_n for $n \geq 1$ has a leading z^{-n} singularity. A regular solution on a disk therefore uses J_n only. An annulus need not exclude Y_n , because its physical domain does not contain $r = 0$.

Useful recurrences, obtained by differentiating the series, are

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z), \quad (31.3)$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z). \quad (31.4)$$

In particular, $J'_0 = -J_1$. At large positive z ,

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right), \quad Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right).$$

The amplitude decreases as $z^{-1/2}$, unlike a Cartesian sine wave.

31.3 Orthogonality on a finite disk

The radial equation can be put in Sturm–Liouville form:

$$-(rR')' + \frac{m^2}{r}R = k^2 rR.$$

Thus the weight is r , exactly the radial factor in the area element $r dr d\phi$. Let $R_n(r) = J_m(k_n r)$ and impose regularity at zero and Dirichlet data $R_n(a) = 0$. If $j_{m,n}$ is the n th positive zero of J_m , then $k_n = j_{m,n}/a$.

Subtracting the two radial eigenvalue equations gives

$$(k_n^2 - k_p^2) \int_0^a r R_n R_p dr = [r(R_n R_p' - R_p R_n')]_0^a = 0.$$

Hence different radial modes are orthogonal. For the norm, Bessel's recurrences give the integral identity

$$\int_0^a r J_m(kr)^2 dr = \frac{a^2}{2} [J_m(ka)^2 - J_{m-1}(ka)J_{m+1}(ka)].$$

At a zero $j_{m,n}$, the first recurrence gives $J_{m-1}(j_{m,n}) = -J_{m+1}(j_{m,n})$. Consequently,

$$\int_0^a r J_m(j_{m,n}r/a) J_m(j_{m,p}r/a) dr = \frac{a^2}{2} J_{m+1}(j_{m,n})^2 \delta_{np}. \quad (31.5)$$

An admissible radial function therefore has the Fourier–Bessel expansion

$$f(r) = \sum_{n=1}^{\infty} c_n J_m(j_{m,n}r/a),$$

with

$$c_n = \frac{2}{a^2 J_{m+1}(j_{m,n})^2} \int_0^a r f(r) J_m(j_{m,n}r/a) dr.$$

For a Neumann boundary condition, the allowed wavenumbers instead satisfy $J_m'(k_n a) = 0$. A zero-frequency constant mode must be included when $m = 0$ and both regularity and a Neumann condition are imposed.

31.4 Modified and spherical Bessel functions

If radial separation produces the equation

$$z^2 y'' + zy' - (z^2 + \nu^2)y = 0,$$

its solutions are the modified Bessel functions I_ν and K_ν . The regular solution is

$$I_\nu(z) = \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(j + \nu + 1)} \left(\frac{z}{2}\right)^{2j+\nu}.$$

For positive large z ,

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \quad K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}.$$

Thus I_ν is chosen for regular interior solutions and K_ν for radially decaying exterior solutions when the corresponding endpoint conditions apply.

The three-dimensional radial Helmholtz equation is

$$R'' + \frac{2}{r}R' + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]R = 0.$$

Putting $R(r) = r^{-1/2}y(kr)$ gives Bessel's equation of order $\nu = \ell + \frac{1}{2}$. The spherical Bessel functions are

$$j_\ell(z) = \sqrt{\frac{\pi}{2z}} J_{\ell+1/2}(z), \quad y_\ell(z) = \sqrt{\frac{\pi}{2z}} Y_{\ell+1/2}(z).$$

For example,

$$j_0(z) = \frac{\sin z}{z}, \quad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}, \quad y_0(z) = -\frac{\cos z}{z}.$$

For an unbounded radial domain, the discrete Fourier–Bessel expansion is replaced by the Hankel transform; its measure and normalization are derived in [Section 25](#).

Part VII

Partial Differential Equations

Partial
differential
equations

32 Differential equations, classification, and data

32.1 From ordinary to partial differential equations

An ordinary differential equation describes a function of one independent variable. A partial differential equation describes a field depending on several independent variables, such as a concentration $u(x, t)$, a membrane displacement $u(x, y, t)$, or an electrostatic potential $\phi(x, y, z)$.

An equation is *linear* when the unknown function and all its derivatives occur linearly, and their coefficients do not depend on the unknown. For example,

$$y''' - 3y'' - 2x^2y' = 7 \sin x$$

is linear even though its coefficients depend on x . In contrast, $y' = y^2$ is nonlinear. For a linear operator L , the equation $Ly = 0$ is homogeneous and $Ly = f$ is inhomogeneous. The term f is often called a source, forcing, or driving term. Constant coefficients and homogeneity are separate properties: $y' + 3xy = 0$ is homogeneous but has a variable coefficient, whereas $y' + 3y = x^2$ has constant coefficients but is inhomogeneous.

The *order* of a differential equation is the order of its highest derivative. For a regular linear ODE of order n , the homogeneous solution space has dimension n . Thus

$$y(x) = y_p(x) + \sum_{j=1}^n C_j y_j(x),$$

where y_p is one particular solution and $y_1, \dots, y_n$ are independent homogeneous solutions. Initial values of $y, y', \dots, y^{(n-1)}$ at a regular point specify the n constants. Boundary-value conditions must also be checked for compatibility and uniqueness.

For a constant-coefficient homogeneous ODE, substitute $y = e^{rx}$ to obtain its characteristic polynomial. A simple root r gives e^{rx} ; a root of multiplicity q gives

$$e^{rx}, \quad xe^{rx}, \quad \dots, \quad x^{q-1}e^{rx}.$$

Complex-conjugate roots give real sine and cosine combinations. The superposition principle survives for linear PDEs, but their solution spaces are generally infinite dimensional. Consequently their initial and boundary data are functions, rather than a finite list of numbers.

32.2 The principal part of a second-order equation

In two variables, consider

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G. \quad (32.1)$$

The terms containing second derivatives form the *principal part*. Its symmetric coefficient matrix is

$$M = \begin{pmatrix} A & B \\ B & C \end{pmatrix}.$$

The type is determined by

$$\Delta = B^2 - AC = -\det M.$$

The three cases are:

Type	Condition	Standard model
Hyperbolic	$\Delta > 0$	$u_{tt} - c^2 u_{xx} = 0$
Parabolic	$\Delta = 0$	$u_t - Du_{xx} = 0$
Elliptic	$\Delta < 0$	$u_{xx} + u_{yy} = 0$

If the mixed derivative is instead written bu_{xy} , its discriminant is $b^2 - 4AC$. This is the same test because $b = 2B$.

A real rotation diagonalizes M . In the hyperbolic case its two eigenvalues have opposite signs, so the canonical principal part is a difference of squares. In the elliptic case the eigenvalues have the same sign, so the canonical principal part is a sum of squares. In the parabolic case one eigenvalue is zero and the other is nonzero; the missing second derivative is replaced by a first derivative in the standard diffusion equation. If all second-derivative coefficients vanish, the equation has lower order and should be analyzed as such. For variable coefficients, the type may vary from point to point.

The classification concerns the principal part. Lower-order terms affect growth, decay, forcing, and boundary behavior, but do not change this local type unless the principal part itself becomes degenerate.

32.3 Hyperbolic equations and Cauchy data

The one-dimensional wave equation is

$$u_{tt} = c^2 u_{xx}.$$

Here t is a time variable and x is a space variable. If spatial dependence is removed, the equation reduces to $u_{tt} = 0$, the free Newtonian equation. Its solution requires an initial position and velocity. The field equation likewise requires two initial functions:

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \quad (32.2)$$

These are Cauchy data on the noncharacteristic curve $t = 0$. Data on a characteristic curve require special treatment because the PDE does not then determine all derivatives transverse to the curve. On a finite spatial interval, one also specifies suitable endpoint conditions, such as fixed-end or free-end conditions. Wave characteristics and the resulting solution formulas are derived in Section 40.

32.4 Parabolic equations and initial-boundary data

The diffusion equation

$$u_t = Du_{xx}, \quad D > 0,$$

is first order in time and second order in space. It therefore requires one initial function, $u(x, 0) = f(x)$. On $0 < x < L$, it also requires suitable spatial boundary conditions, for example

$$u(0, t) = a(t), \quad u(L, t) = b(t)$$

for Dirichlet data, or prescribed normal derivatives for Neumann data. The initial values and boundary values must agree at the corners when a smooth solution up to $t = 0$ is desired. A prescribed initial time derivative is not independent data: it is fixed by the PDE through $u_t(x, 0) = Df''(x)$ whenever these derivatives exist. The full diffusion solution is developed in Section 38.

32.5 Elliptic equations and boundary-value problems

Laplace's equation,

$$\nabla^2 \phi = 0,$$

and Poisson's equation,

$$-\nabla^2 \phi = f,$$

describe spatial boundary-value problems. On a bounded region Ω , Dirichlet data prescribe ϕ on the boundary, while Neumann data prescribe its outward normal derivative $\partial_n \phi = \nabla \phi \cdot \mathbf{n}$. Robin data prescribe a combination $a\phi + b\partial_n \phi$.

For a pure Neumann problem $-\nabla^2 \phi = f$ with $\partial_n \phi = g$, the divergence theorem imposes

$$\int_{\Omega} f \, dV = - \int_{\partial\Omega} g \, dS.$$

Moreover, adding a constant to ϕ leaves both the equation and Neumann data unchanged, so uniqueness requires fixing an additive constant. Specifying both an arbitrary function and an arbitrary normal derivative on an entire closed boundary generally overdetermines an elliptic problem.

33 Conservation laws and constitutive relations

33.1 From integral balance to the continuity equation

Let $u(\mathbf{x}, t)$ be the density of a quantity: mass, particle number, charge, heat, or probability. Its amount in a fixed subregion R is

$$U_R(t) = \int_R u(\mathbf{x}, t) \, dV.$$

Let $\mathbf{J}$ be its flux, defined so that $\mathbf{J} \cdot \mathbf{n} \, dS$ is the amount crossing a small surface element per unit time in the direction $\mathbf{n}$. Let Q be the production rate per unit volume, positive for a source and negative for a sink.

Conservation on the fixed region gives

$$\frac{d}{dt} \int_R u \, dV = - \int_{\partial R} \mathbf{J} \cdot \mathbf{n} \, dS + \int_R Q \, dV. \quad (33.1)$$

The minus sign appears because the normal is outward: positive outward flux removes the quantity from R . Apply the divergence theorem and move the time derivative inside the integral:

$$\int_R (u_t + \nabla \cdot \mathbf{J} - Q) \, dV = 0.$$

Because this holds for every subregion, a sufficiently regular integrand must vanish pointwise:

$$u_t + \nabla \cdot \mathbf{J} = Q. \quad (33.2)$$

This is the local conservation or continuity equation.

In one dimension,

$$\frac{d}{dt} \int_{x_1}^{x_2} u(x, t) \, dx = J(x_1, t) - J(x_2, t) + \int_{x_1}^{x_2} Q(x, t) \, dx.$$

A conservation law does not require the amount in every interval to be constant. It requires any change to be accounted for by boundary flux and sources. With $Q = 0$ and equal endpoint fluxes, the total is constant. With a decay term $Q = -\alpha u$, $\alpha > 0$, the total can decrease even when there is no net boundary flux.

A vector-valued conserved quantity has a tensor-valued flux. For example, if p_i is momentum density and Π_{ij} is its flux tensor, then

$$\partial_i p_i + \partial_j \Pi_{ij} = f_i.$$

The index i labels the momentum component and j the direction of its transport. Repeated Cartesian indices are summed.

33.2 Boundary conditions and global conservation

A no-flux boundary condition is

$$\mathbf{J} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \quad (33.3)$$

It prohibits transport through the wall. It does not require the tangential component of $\mathbf{J}$ to vanish. With $Q = 0$, integration over the whole domain gives

$$\frac{d}{dt} \int_{\Omega} u \, dV = 0.$$

The mathematical boundary condition depends on the constitutive law for $\mathbf{J}$. For diffusion, $\mathbf{J} = -D\nabla u$ with $D > 0$, so no flux means $\partial_n u = 0$, a homogeneous Neumann condition. Prescribing u itself on the boundary is a Dirichlet condition and usually permits exchange with the surroundings. The resulting flux is whatever is required to maintain the prescribed boundary value.

33.3 Transport and diffusion

For material carried with velocity $\mathbf{v}$, the advective flux is

$$\mathbf{J} = u\mathbf{v}.$$

Its dimensions are density times velocity. In one dimension with constant velocity c and no source,

$$u_t + (cu)_x = u_t + cu_x = 0.$$

For a variable velocity field,

$$u_t + \mathbf{v} \cdot \nabla u + u \nabla \cdot \mathbf{v} = 0.$$

Thus pure conservation of density includes compression or dilation when $\nabla \cdot \mathbf{v} \neq 0$.

Diffusion transports material down a density gradient. Fick's law is

$$\mathbf{J} = -D\nabla u,$$

where D has dimensions length squared per time. Substitution into (33.2) gives

$$u_t = D\nabla^2 u + Q$$

for constant D . For spatially variable diffusivity the correct expression is $u_t = \nabla \cdot (D\nabla u) + Q$. Fourier's law of heat conduction has the same gradient structure after the heat capacity is included.

33.4 Reaction–diffusion and chemotaxis

If the source depends on the local density, $Q = f(u)$, then

$$u_t = D\nabla^2 u + f(u).$$

For example, logistic production gives $f(u) = ru(1 - u/K)$, where r is a growth rate and K a carrying density. In the absence of spatial variation this reduces to the logistic ODE.

Chemotaxis describes motion in response to a chemical gradient. Let u be the organism density and c the chemoattractant concentration. A model with a prescribed flux magnitude M and motion toward increasing c is

$$\mathbf{J}_c = M \frac{\nabla c}{|\nabla c|}, \quad \nabla c \neq 0.$$

Combining this with ordinary diffusion $\mathbf{J}_d = -D\nabla u$ gives

$$u_t = D\nabla^2 u - M\nabla \cdot \left(\frac{\nabla c}{|\nabla c|} \right).$$

The direction is undefined when $\nabla c = 0$, so a model using this law must prescribe its behavior there, for example by a smooth regularization. A common density-dependent

alternative takes the chemotactic velocity to be $\chi \nabla c$, giving $\mathbf{J}_c = \chi u \nabla c$. If the cells produce the attractant, a coupled model is

$$\begin{aligned} u_t &= D_u \nabla^2 u - \nabla \cdot (\chi u \nabla c), \\ c_t &= D_c \nabla^2 c + \alpha u - \beta c. \end{aligned}$$

This is a Keller–Segel type system. The first equation conserves cell number under no-flux conditions; the second contains production and decay of the chemical.

33.5 Thin-film flow and higher-order fluxes

For a thin viscous film, the height $h(\mathbf{x}, t)$ is a conserved volume density per unit substrate area. In a capillary model, the pressure is $p = -\gamma \nabla^2 h$, and the mobility is proportional to h^3 . Choosing units that absorb the positive material constants gives

$$\mathbf{J} = -h^3 \nabla p = h^3 \nabla \nabla^2 h.$$

Conservation then gives the fourth-order equation

$$h_t = -\nabla \cdot (h^3 \nabla \nabla^2 h). \quad (33.4)$$

The no-flux condition is $h^3 \partial_n \nabla^2 h = 0$. If $h > 0$ at the boundary this reduces to $\partial_n \nabla^2 h = 0$; a dry edge $h = 0$ also makes the flux vanish. A fourth-order spatial equation needs additional boundary data, such as an appropriate contact-angle condition. No flux alone does not supply a complete boundary-value problem.

33.6 Waves as momentum conservation

For a stretched string or membrane, the displacement u is not itself the conserved density. Momentum density is proportional to u_t . In a scaled model, let the momentum flux be $-c^2 \nabla u$. Its conservation law is

$$\partial_t(u_t) + \nabla \cdot (-c^2 \nabla u) = 0,$$

which gives

$$u_{tt} = c^2 \nabla^2 u.$$

The second time derivative appears because the conserved quantity is a velocity-related field rather than the displacement.

33.7 Steady states and Poisson's equation

A steady state is independent of time, so the continuity equation becomes

$$\nabla \cdot \mathbf{J} = Q.$$

This does not require zero flux. A nonzero flux can balance a source or can carry material through the region while maintaining a time-independent density. For diffusion,

$$-D \nabla^2 u = Q.$$

This is Poisson's equation. When $Q = 0$, it reduces to Laplace's equation. Boundary data determine which steady state is realized. For example, a one-dimensional interval held at two different fixed temperatures has a linear steady temperature profile and a nonzero constant heat flux.

34 Conserved and dissipated functionals

34.1 What a functional measures

A functional assigns a number to an entire function. Examples include

$$M[u] = \int_{\Omega} u \, dV, \quad F[u] = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dV, \quad H[u] = \int_{\Omega} u^4 \, dV.$$

For a time-dependent solution, each becomes a function of time. A functional is conserved if $dF/dt = 0$ and dissipated if $dF/dt \leq 0$. Such statements can reveal stability or decay without first finding the complete solution.

The following calculations assume enough regularity to differentiate under the integral and integrate by parts. They also show exactly which boundary terms must vanish.

34.2 Energy conservation for a string

Consider

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad u(0, t) = u(L, t) = 0.$$

Define

$$E(t) = \frac{1}{2} \int_0^L (u_t^2 + c^2 u_x^2) \, dx.$$

The two terms are the scaled kinetic and elastic energies. Differentiate:

$$\begin{aligned} \frac{dE}{dt} &= \int_0^L (u_t u_{tt} + c^2 u_x u_{xt}) \, dx \\ &= c^2 [u_x u_t]_0^L + \int_0^L u_t (u_{tt} - c^2 u_{xx}) \, dx. \end{aligned}$$

The PDE makes the integral zero. Since the endpoint displacements are fixed for every t , their time derivatives also vanish. Therefore

$$\frac{dE}{dt} = 0. \tag{34.1}$$

The same conclusion holds for homogeneous free-end conditions $u_x = 0$ or suitable periodic conditions because they also eliminate the boundary energy flux.

For the fixed-end problem, $E = 0$ implies $u_t = u_x = 0$ and hence $u = 0$. Thus zero initial displacement and velocity remain zero. Applying this argument to the difference of two solutions proves uniqueness for the same initial and boundary data. If the initial energy is positive, the system cannot dissipate all its motion and deformation to the zero state while conserving energy. Energy conservation alone does not say that the displacement at each individual point remains nonzero at every instant.

34.3 Dissipation for diffusion

Let

$$u_t = Du_{xx}, \quad u(0, t) = u(L, t) = 0, \quad D > 0.$$

For the squared-amplitude functional $N(t) = \frac{1}{2} \int_0^L u^2 dx$,

$$\begin{aligned} \frac{dN}{dt} &= D \int_0^L uu_{xx} dx \\ &= D[uu_x]_0^L - D \int_0^L u_x^2 dx = -D \int_0^L u_x^2 dx \leq 0. \end{aligned}$$

Thus the amplitude decreases in the L^2 sense.

Now use the gradient functional

$$F(t) = \frac{1}{2} \int_0^L u_x^2 dx.$$

Differentiating and integrating by parts gives

$$\begin{aligned} \frac{dF}{dt} &= \int_0^L u_x u_{xt} dx \\ &= [u_x u_t]_0^L - \int_0^L u_{xx} u_t dx \\ &= -D \int_0^L u_{xx}^2 dx \leq 0. \end{aligned}$$

Again, $u_t = 0$ at a fixed Dirichlet boundary removes the boundary term. The derivative need not be strictly negative: it is zero for a stationary solution.

For a graph $y = u(x)$ with small slope, its arc length is

$$\int_0^L \sqrt{1 + u_x^2} dx = L + \frac{1}{2} \int_0^L u_x^2 dx + O\left(\int_0^L u_x^4 dx\right).$$

Therefore F measures the leading excess arc length above a flat profile. Its decrease explains the smoothing action of diffusion. In contrast, with no-flux boundaries the mass $\int u dx$ is conserved even though these other functionals decay. Conservation and dissipation can describe different quantities in the same system.

34.4 Capillary energy in the thin-film model

The thin-film equation also dissipates a natural gradient energy. Define

$$E[h] = \frac{1}{2} \int_{\Omega} |\nabla h|^2 dV, \quad \mu = \frac{\delta E}{\delta h} = -\nabla^2 h.$$

Equation (33.4) can be written $h_t = \nabla \cdot (h^3 \nabla \mu)$. Under periodic conditions, or boundary conditions that remove both integration-by-parts terms,

$$\begin{aligned} \frac{dE}{dt} &= \int_{\Omega} \mu h_t dV = \int_{\Omega} \mu \nabla \cdot (h^3 \nabla \mu) dV \\ &= - \int_{\Omega} h^3 |\nabla \mu|^2 dV \leq 0 \quad (h \geq 0). \end{aligned}$$

At the same time, no flux conserves $\int_{\Omega} h dV$. The film redistributes its fixed volume while reducing its capillary energy.

35 First-order linear PDEs and characteristics

35.1 The characteristic equations

Consider

$$a(x, t)u_x + b(x, t)u_t + c(x, t)u = g(x, t). \quad (35.1)$$

The coefficients and source are specified functions. The idea is to follow curves in the (x, t) plane along which the directional derivative equals $a\partial_x + b\partial_t$.

Choose a parameter s and solve

$$\frac{dx}{ds} = a(x, t), \quad \frac{dt}{ds} = b(x, t). \quad (35.2)$$

Along such a curve, define $U(s) = u(x(s), t(s))$. The chain rule gives

$$\frac{dU}{ds} = u_x \frac{dx}{ds} + u_t \frac{dt}{ds} = au_x + bu_t.$$

Thus the PDE becomes the linear ODE

$$U'(s) + c(x(s), t(s))U(s) = g(x(s), t(s)). \quad (35.3)$$

If $U(0) = U_0$, introduce

$$C(s) = \int_0^s c(x(\sigma), t(\sigma)) d\sigma.$$

Multiplying by the integrating factor $e^{C(s)}$ and integrating gives

$$U(s) = e^{-C(s)} \left[U_0 + \int_0^s e^{C(\sigma)} g(x(\sigma), t(\sigma)) d\sigma \right]. \quad (35.4)$$

The task is therefore to find the curves, solve this ODE on each curve, and then express the characteristic labels in terms of x, t .

With initial data $u(x, 0) = f(x)$, label a characteristic by its starting point ξ :

$$x(0, \xi) = \xi, \quad t(0, \xi) = 0, \quad U(0, \xi) = f(\xi).$$

The map $(s, \xi) \mapsto (x, t)$ supplies characteristic coordinates wherever its Jacobian is nonzero.

35.2 Noncharacteristic data and the role of regularity

For initial data on a general curve $(x_0(\xi), t_0(\xi))$, the tangent to that curve must be independent of the characteristic direction. The condition is

$$x'_0(\xi)b(x_0, t_0) - t'_0(\xi)a(x_0, t_0) \neq 0. \quad (35.5)$$

For the horizontal initial line $t = 0$, this reduces to $b(x, 0) \neq 0$. If b vanishes elsewhere, the parameter t may cease to be convenient, but the parameterized system (35.2) can still be regular. A zero of b alone does not force distinct characteristics to intersect.

If a, b are locally Lipschitz and the vector field is nonzero, uniqueness of the characteristic ODE prevents two distinct integral curves from crossing in the region where the flow is regular. A linear PDE therefore has a characteristic geometry determined by its coefficients, independent of the values of the initial function. Closed-form formulas are not always available, but the reduction to ODEs remains valid locally. Degenerate vector fields, characteristic initial curves, or failure of the coordinate map require separate analysis.

Example 35.1 (Incompatible data on a characteristic line). Consider $u_x + u = 0$ with $u(x, 0) = \sin x$. The characteristic direction is horizontal, so $t = 0$ itself is characteristic. Restricting the PDE to the initial line would require

$$\cos x + \sin x = 0$$

for every x , which is false. No differentiable solution satisfies these data. If the data were $u(x, 0) = Ae^{-x}$, they would be compatible, but would still not determine the values on other horizontal characteristics. Indeed $u(x, t) = F(t)e^{-x}$ is a solution for any F with $F(0) = A$.

35.3 Constant-speed transport in characteristic coordinates

For

$$u_t + \alpha u_x = 0, \quad u(x, 0) = f(x),$$

choose $s = t$ and solve $dx/ds = \alpha$. Then

$$x = \xi + \alpha s, \quad t = s, \quad \xi = x - \alpha t.$$

Write $u(x, t) = V(\xi, s)$. By the chain rule,

$$u_t = -\alpha V_\xi + V_s, \quad u_x = V_\xi.$$

The PDE becomes $V_s = 0$. Hence V depends only on ξ , and the initial condition fixes that dependence:

$$u(x, t) = f(x - \alpha t). \quad (35.6)$$

A feature initially at $x = \xi$ moves to $x = \xi + \alpha t$. Positive α carries the profile to the right, negative α to the left. In an (x, t) plot, the characteristic slope is $dt/dx = 1/\alpha$ when $\alpha \neq 0$, whereas its physical speed is $dx/dt = \alpha$.

35.4 Transport with forcing and decay

For constant α and λ , consider

$$u_t + \alpha u_x + \lambda u = g(x, t), \quad u(x, 0) = f(x).$$

The characteristics are unchanged: $x = \xi + \alpha t$. Along each one,

$$U' + \lambda U = g(\xi + \alpha t, t).$$

The integrating factor $e^{\lambda t}$ gives

$$u(x, t) = e^{-\lambda t} f(x - \alpha t) + \int_0^t e^{-\lambda(t-\tau)} g(x - \alpha(t - \tau), \tau) d\tau. \quad (35.7)$$

The first term transports and damps the initial profile. The integral adds the source encountered along the backward characteristic ending at (x, t) .

Example 35.2 (A source proportional to position). Solve

$$u_x + u_t + u = x, \quad u(x, 0) = e^{-x^2}.$$

Here $x = \xi + s$ and $t = s$. The value on a characteristic obeys

$$U' + U = s + \xi.$$

Multiplication by e^s gives

$$\frac{d}{ds}(e^s U) = e^s(s + \xi).$$

Since an antiderivative of the right-hand side is $e^s(s + \xi - 1)$,

$$U(s) = s + \xi - 1 + C(\xi)e^{-s}.$$

The initial value $U(0) = e^{-\xi^2}$ gives $C(\xi) = e^{-\xi^2} - \xi + 1$. Substitute $\xi = x - t$, $s = t$:

$$u(x, t) = e^{-(x-t)^2-t} + (1 - x + t)e^{-t} + x - 1.$$

At $t = 0$, the last two linear terms cancel, leaving the prescribed Gaussian.

Example 35.3 (Variable characteristic speed). Solve

$$xu_x + u_t + 2u = 0, \quad u(x, 0) = \cos x.$$

The characteristic equations are

$$x' = x, \quad t' = 1,$$

so $x = \xi e^s$ and $t = s$. Along them, $U' + 2U = 0$, giving $U(s) = \cos \xi e^{-2s}$. Since $\xi = xe^{-t}$,

$$u(x, t) = e^{-2t} \cos(xe^{-t}).$$

The amplitude decays and the spatial pattern stretches because material points move with velocity proportional to their position.

36 Worked first-order PDE practice

The following problems develop the characteristic calculations explicitly. In each case the initial line is $t = 0$.

36.1 Sine profiles with four transport laws

For $u_t + 9u_x = 0$ and $u(x, 0) = \sin x$, characteristics satisfy $x = \xi + 9t$ and $U' = 0$. Thus

$$u(x, t) = \sin(x - 9t).$$

For $u_t - 8u_x = 0$, the speed is -8 and $x = \xi - 8t$, giving

$$u(x, t) = \sin(x + 8t).$$

For $u_t + \pi u_x = 0$, the same reasoning gives

$$u(x, t) = \sin(x - \pi t).$$

Finally, for $u_t + \pi u_x + u = 0$, the curves are still $x = \xi + \pi t$, but $U' + U = 0$. The initial amplitude is $\sin \xi$, so

$$u(x, t) = e^{-t} \sin(x - \pi t).$$

The zero-order term changes the amplitude along a characteristic, not its trajectory.

36.2 Constant forcing with quadratic initial data

Solve $u_t + 3u_x = 1$, $u(x, 0) = x^2$. On $x = \xi + 3t$,

$$\frac{dU}{dt} = 1, \quad U(0) = \xi^2.$$

Hence $U = \xi^2 + t$ and

$$u(x, t) = (x - 3t)^2 + t.$$

The quadratic shape moves at speed three while the source adds the same amount t to every characteristic.

36.3 Position-dependent forcing with exponential data

Solve $u_t + 3u_x = x$, $u(x, 0) = e^x$. Along $x = \xi + 3t$,

$$U' = \xi + 3t.$$

Integrating from 0 to t gives

$$U(t) = e^\xi + \xi t + \frac{3}{2}t^2.$$

Substitution of $\xi = x - 3t$ gives

$$u(x, t) = e^{x-3t} + xt - \frac{3}{2}t^2.$$

The forcing must be integrated at the moving position $\xi + 3t$; treating x as constant during that characteristic integral would give a different and incorrect result.

36.4 A position-dependent decay coefficient

Solve $u_x + u_t + xu = 0$, $u(x, 0) = \cos x$. The curves are $x = \xi + t$. The amplitude satisfies

$$U' + (\xi + t)U = 0.$$

Integration gives

$$U(t) = \cos \xi \exp \left(-\xi t - \frac{1}{2}t^2 \right).$$

Using $\xi = x - t$,

$$u(x, t) = \cos(x - t) \exp \left(-xt + \frac{1}{2}t^2 \right).$$

The sign of x affects whether the zero-order term locally damps or amplifies the transported value.

36.5 Rescaling the PDE changes the parameter, not the solution

Compare

$$u_x + u_t + u = 1 \quad \text{and} \quad 2u_x + 2u_t + 2u = 2, \quad u(x, 0) = \cos x.$$

For the first equation, use $x = \xi + s$, $t = s$. Then $U' + U = 1$ and

$$U(s) = 1 + (\cos \xi - 1)e^{-s}.$$

For the second equation, use $x = \xi + 2s$, $t = 2s$. Now $U' + 2U = 2$, and

$$U(s) = 1 + (\cos \xi - 1)e^{-2s}.$$

In both cases $\xi = x - t$; in the second, $s = t/2$. Therefore both give

$$u(x, t) = 1 + [\cos(x - t) - 1]e^{-t}.$$

Multiplying the entire PDE by a nonzero constant does not change the physical characteristic curves. It only changes how rapidly the parameter s moves along them.

36.6 Zero initial data with complicated coefficients

Consider

$$(1 + x^2)u_t + x^2u_x + e^xu = 0, \quad u(x, 0) = 0.$$

The coefficient $1 + x^2$ never vanishes, so divide by it and use t as the characteristic parameter:

$$\frac{dx}{dt} = \frac{x^2}{1 + x^2}, \quad \frac{dU}{dt} + \frac{e^{x(t)}}{1 + x(t)^2}U = 0.$$

The initial amplitude is zero on every characteristic. The unique solution of this homogeneous linear ODE with zero initial value is $U = 0$. Therefore

$$u(x, t) = 0.$$

There is no need to invert the characteristic curve explicitly. This illustrates why checking the structure of the amplitude equation can save unnecessary calculation.

36.7 Two more translated profiles

For $u_t - 5u_x = 0$ with $u(x, 0) = (1 + x^2)^{-1}$, the characteristics are $x = \xi - 5t$ and U is constant. Hence

$$u(x, t) = \frac{1}{1 + (x + 5t)^2}.$$

For $u_t + 2u_x = 0$ with $u(x, 0) = \cos x$, they are $x = \xi + 2t$, so

$$u(x, t) = \cos(x - 2t).$$

The translation rule applies to the complete initial function, not merely to sinusoidal data.

36.8 Time-dependent damping

Solve $u_x + u_t + tu = 0$, $u(x, 0) = \cos x$. Since $x = \xi + t$,

$$U' + tU = 0.$$

Its integrating factor is $e^{t^2/2}$, so

$$U(t) = \cos \xi e^{-t^2/2}, \quad u(x, t) = e^{-t^2/2} \cos(x - t).$$

The attenuation is Gaussian in time because the decay rate itself grows linearly with time.

36.9 Constant forcing with linear data

Solve $u_x + u_t = 5$, $u(x, 0) = x$. Along $x = \xi + t$, $U' = 5$ and $U(0) = \xi$, so

$$U = \xi + 5t, \quad u(x, t) = x + 4t.$$

Direct substitution gives $u_x + u_t = 1 + 4 = 5$, confirming the result.

37 Nonlinear conservation laws, traffic, and shocks

37.1 Quasilinear equations and data-dependent characteristics

A first-order quasilinear equation has the form

$$a(x, t, u)u_x + b(x, t, u)u_t = q(x, t, u).$$

It is linear in the first derivatives but its coefficients may depend on u . Its characteristic system is now coupled:

$$\frac{dx}{ds} = a(x, t, U), \quad \frac{dt}{ds} = b(x, t, U), \quad \frac{dU}{ds} = q(x, t, U).$$

The curves cannot generally be found independently of the unknown values of the solution.

A particularly important example is the scalar conservation law

$$u_t + \partial_x J(u) = 0. \quad (37.1)$$

For a differentiable solution, this is $u_t + J'(u)u_x = 0$. Taking t as the parameter gives

$$\frac{dx}{dt} = J'(U), \quad \frac{dU}{dt} = 0.$$

With initial data $u(x, 0) = u_0(x)$,

$$U(t, \xi) = u_0(\xi), \quad x(t, \xi) = \xi + J'(u_0(\xi))t. \quad (37.2)$$

Each characteristic is a straight line, but its speed depends on its initial value. Unlike the constant-speed transport problem, different characteristics need not be parallel.

37.2 When a smooth solution breaks down

Differentiate the characteristic map with respect to ξ :

$$\frac{\partial x}{\partial \xi} = 1 + tJ''(u_0(\xi))u'_0(\xi).$$

As long as this is nonzero, the map can be locally inverted. By the chain rule,

$$u_x(x, t) = \frac{u'_0(\xi)}{1 + tJ''(u_0(\xi))u'_0(\xi)}. \quad (37.3)$$

If the denominator reaches zero while the numerator does not, the spatial gradient diverges although u itself remains finite. If

$$m_* = \inf_{\xi} J''(u_0(\xi))u'_0(\xi) < 0,$$

and the minimum is attained in the relevant smooth data, the first breaking time is

$$t_* = -\frac{1}{m_*}.$$

Continuing the intersecting characteristic lines beyond t_* would assign more than one value to the same (x, t) . A classical single-valued smooth solution no longer exists there. The conservation law must instead be interpreted weakly, or a small physical regularizing term must be retained.

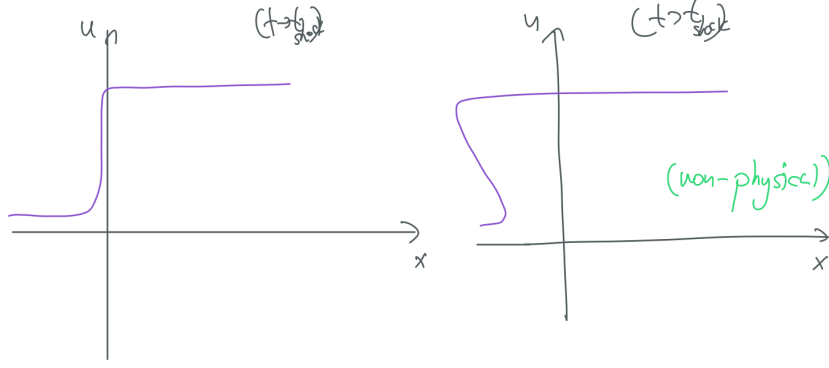


Figure 18: An inviscid profile steepens and its formal characteristic continuation becomes multivalued.

37.3 The traffic-flow constitutive law

Let $u(x, t)$ be car density on a one-way road and $v(u)$ the average car speed. The number of cars crossing a point per unit time is

$$J(u) = u v(u).$$

Conservation supplies the PDE, while a constitutive law supplies $v(u)$. A simple decreasing-speed model is

$$v(u) = v_{\max} \left(1 - \frac{u}{u_{\max}} \right), \quad 0 \leq u \leq u_{\max}.$$

Here $u_{\max}$ is bumper-to-bumper density. It follows that

$$\begin{aligned} J(u) &= v_{\max} u \left(1 - \frac{u}{u_{\max}} \right), \\ J'(u) &= v_{\max} \left(1 - \frac{2u}{u_{\max}} \right), \\ J''(u) &= -\frac{2v_{\max}}{u_{\max}} < 0. \end{aligned}$$

The characteristic speed $J'(u)$ differs from the car speed $v(u)$. Cars move forward, but a density disturbance can travel backward when $u > u_{\max}/2$. This is the familiar upstream movement of a traffic jam.

Suppose the initial density increases with position, from freely flowing traffic behind to congested traffic ahead. Then $u'_0 > 0$, while $J'' < 0$. The faster characteristics from the less dense region catch the slower ones ahead. Equation (37.3) predicts steepening and eventual shock formation. For this linear speed-density law,

$$t_* = \frac{u_{\max}}{2v_{\max} \max_{\xi} u'_0(\xi)}$$

when the maximum positive slope exists. An initial decreasing density profile makes neighboring characteristics spread instead; it produces rarefaction rather than compression.

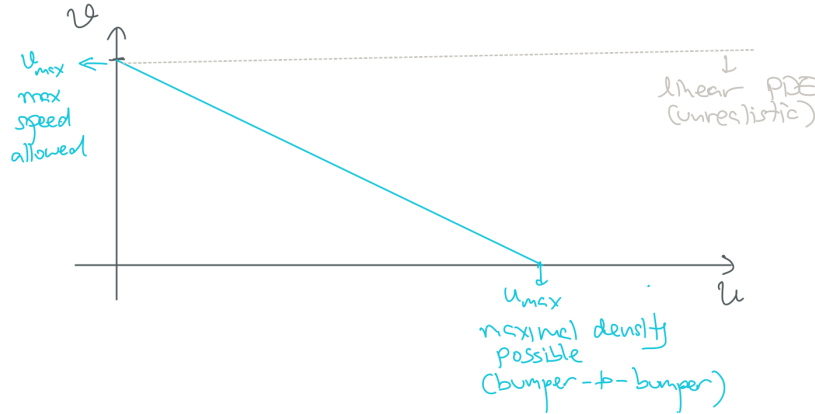


Figure 19: A decreasing constitutive relation between car speed and traffic density.

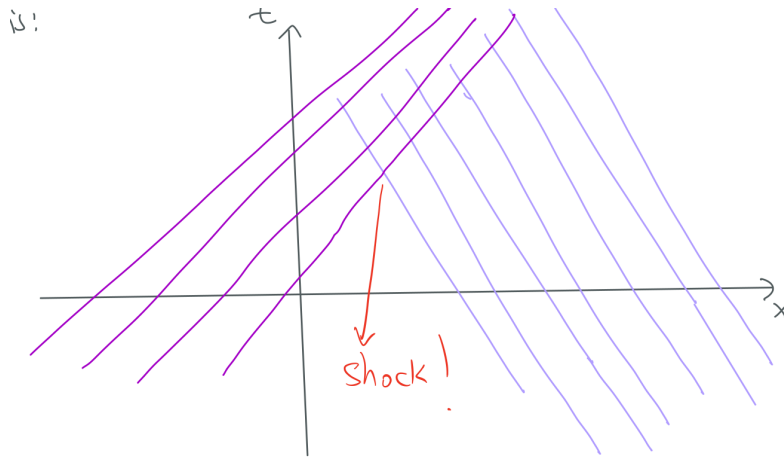


Figure 20: Characteristic lines converge as traffic density increases toward a congested region.

37.4 Conservation across a moving discontinuity

A shock is a moving discontinuity at $x = s(t)$, with limiting values u_L on the left and u_R on the right. Its speed is not chosen by averaging characteristics arbitrarily; it follows from conservation.

Integrate the conservation law over a small interval straddling the moving front, or write

$$u = u_L + (u_R - u_L)H(x - s(t))$$

locally for constant left and right states. The singular terms are

$$u_t = -s'(t)(u_R - u_L)\delta(x - s(t)),$$

$$\partial_x J(u) = [J(u_R) - J(u_L)]\delta(x - s(t)).$$

Their coefficients must cancel. Thus the Rankine–Hugoniot condition is

$$s' = \frac{J(u_R) - J(u_L)}{u_R - u_L}, \quad u_R \neq u_L. \quad (37.4)$$

It expresses equality of flux measured in the frame moving with the front: $J(u_L) - s'u_L = J(u_R) - s'u_R$.

For a physically admissible compressive shock of a scalar conservation law, characteristics enter the front from both sides:

$$J'(u_L) > s' > J'(u_R).$$

For the concave traffic flux this occurs when $u_L < u_R$. In that case

$$s' = v_{\max} \left(1 - \frac{u_L + u_R}{u_{\max}} \right).$$

A sufficiently dense front has negative speed, so the jam moves upstream even though the cars continue to move forward.

37.5 Burgers' equation and rarefaction

For the convex flux $J(u) = u^2/2$, the conservation law becomes the inviscid Burgers equation,

$$u_t + uu_x = 0. \quad (37.5)$$

The characteristic lines are $x = \xi + u_0(\xi)t$, and

$$u_x = \frac{u'_0(\xi)}{1 + tu'_0(\xi)}.$$

A negative initial slope steepens. For a discontinuity with $u_L > u_R$, the shock speed is

$$s' = \frac{u_L + u_R}{2},$$

which lies between u_L and u_R as required.

If instead $u_L < u_R$, the characteristic lines separate. A shock would have characteristics leaving the discontinuity and is not the solution selected by small positive viscosity. Seek a continuous fan depending on $\eta = x/t$, $u(x, t) = U(\eta)$. Then

$$u_t = -\frac{\eta}{t}U', \quad u_x = \frac{1}{t}U',$$

so Burgers' equation becomes $(U - \eta)U' = 0$. The nonconstant part has $U = \eta$. Matching the constant states gives

$$u(x, t) = \begin{cases} u_L, & x/t \leq u_L, \\ x/t, & u_L < x/t < u_R, \\ u_R, & x/t \geq u_R. \end{cases}$$

This rarefaction fan fills the region between the separating families.

37.6 Small diffusion and a finite-width travelling shock

A flux depending only on u neglects small gradient corrections. Include a diffusive term,

$$J_{\text{total}} = J_0(u) - Du_x, \quad D > 0.$$

Conservation becomes

$$u_t + J'_0(u)u_x = Du_{xx}. \quad (37.6)$$

For the quadratic Burgers flux this is the viscous Burgers equation

$$u_t + uu_x = Du_{xx}.$$

Even a small D matters once the inviscid solution develops a large gradient. It smooths a discontinuity over a narrow region.

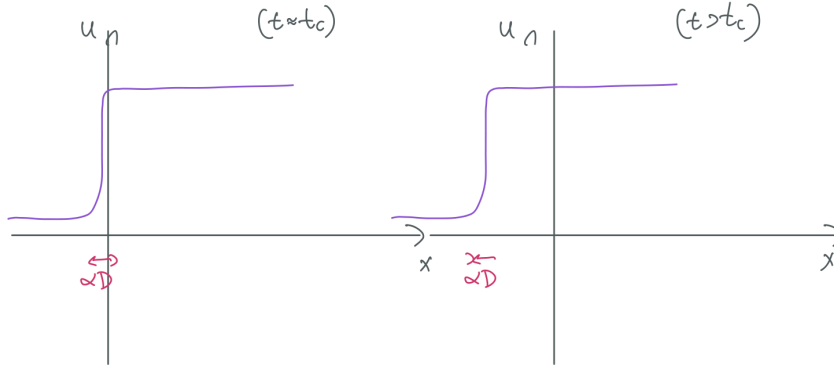


Figure 21: A finite diffusion coefficient replaces an abrupt traffic front by a translating profile of finite width.

To find the profile explicitly, put $u(x, t) = U(z)$ with $z = x - st$. Then

$$(U - s)U' = DU''.$$

Integrate once:

$$DU' = \frac{1}{2}U^2 - sU + C.$$

Assume $U(-\infty) = u_L$, $U(+\infty) = u_R$, and $U' \rightarrow 0$ at both ends. The constants therefore satisfy

$$0 = \frac{1}{2}u_L^2 - su_L + C, \quad 0 = \frac{1}{2}u_R^2 - su_R + C.$$

Subtracting gives $s = (u_L + u_R)/2$, precisely the conservation-law shock speed. The ODE reduces to

$$DU' = \frac{1}{2}(U - u_L)(U - u_R).$$

For $u_L > u_R$, set $a = (u_L - u_R)/2$ and $U = s + w$. Then

$$Dw' = \frac{1}{2}(w^2 - a^2).$$

Separation of variables gives

$$\int \frac{dw}{w^2 - a^2} = \frac{z - z_0}{2D},$$

whose solution matching the two asymptotic states is

$$U(z) = \frac{u_L + u_R}{2} - \frac{u_L - u_R}{2} \tanh \left[\frac{(u_L - u_R)(z - z_0)}{4D} \right]. \quad (37.7)$$

Its thickness is of order $D/(u_L - u_R)$. As $D \rightarrow 0^+$ the profile becomes a sharp shock while retaining the finite propagation speed.

The traffic model can be put into the same form. Write $J_0(u) = \alpha u - \frac{1}{2}\beta u^2$ with $\alpha = v_{\max}$ and $\beta = 2v_{\max}/u_{\max}$. Define the disturbance speed $w = \alpha - \beta u$. Multiplying $u_t + (\alpha - \beta u)u_x = Du_{xx}$ by $-\beta$ gives

$$w_t + ww_x = Dw_{xx}.$$

Thus an increasing traffic-density front corresponds to a decreasing Burgers profile in w . The regularized wave translates with a stable finite width instead of developing a multivalued density.

38 Heat and diffusion equations

38.1 The equation and its initial data

Let $u(x, t)$ be a temperature excess, concentration, or another scalar quantity undergoing linear diffusion. On the infinite line the initial-value problem is

$$u_t = Du_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad u(x, 0) = q(x), \quad D > 0. \quad (38.1)$$

The diffusion coefficient has dimensions $[D] = L^2/T$. An equivalent notation used in the Green-function reading is $\gamma u_t = u_{xx}$, with $\gamma = 1/D$. Because the equation is first order in time, its initial condition specifies u , without independently specifying u_t . The equation itself determines the initial time derivative whenever the initial profile is sufficiently smooth.

The local conservation law and Fick's constitutive law are

$$u_t + J_x = 0, \quad J = -Du_x.$$

Their combination gives (38.1). Integrating over an interval makes the meaning of conservation explicit:

$$\frac{d}{dt} \int_a^b u(x, t) dx = J(a, t) - J(b, t) = D[u_x(b, t) - u_x(a, t)]. \quad (38.2)$$

The total changes only by a flux through the boundary. In particular, $\int_{\mathbb{R}} u dx$ is constant when the flux vanishes at both ends. Conservation of the spatial integral

is compatible with the smoothing of the spatial profile: the quantity is redistributed rather than destroyed.

In d spatial dimensions the corresponding statements are

$$u_t + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -D\nabla u, \quad u_t = D\nabla^2 u. \quad (38.3)$$

For a spatially varying diffusion coefficient, the conservation equation instead gives $u_t = \nabla \cdot (D(\mathbf{x})\nabla u)$; the constant-coefficient Fourier solution below does not directly apply to that operator.

38.2 Fourier transformation reduces the PDE to independent ODEs

Use the spatial transform pair

$$\widehat{u}(k, t) = \int_{\mathbb{R}} u(x, t) e^{-ikx} dx, \quad u(x, t) = \int_{\mathbb{R}} \frac{dk}{2\pi} \widehat{u}(k, t) e^{ikx}.$$

Twice integrating by parts, or differentiating the inverse transform, gives $\widehat{u_{xx}} = -k^2 \widehat{u}$. Consequently,

$$\partial_t \widehat{u}(k, t) = -Dk^2 \widehat{u}(k, t), \quad \widehat{u}(k, 0) = \widehat{q}(k).$$

For each fixed k this is an ordinary differential equation, with solution

$$\widehat{u}(k, t) = \widehat{q}(k) e^{-Dk^2 t}. \quad (38.4)$$

A nonzero Fourier mode decays on the time scale

$$\tau_k = \frac{1}{Dk^2} = \frac{\lambda^2}{4\pi^2 D}, \quad \lambda = \frac{2\pi}{|k|}.$$

Small-scale variations therefore disappear faster than large-scale ones. The $k = 0$ mode does not decay; it is the conserved integral of u .

Substituting the transform of the initial condition into the inverse transform yields

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikx - Dk^2 t} \int_{\mathbb{R}} q(x_0) e^{-ikx_0} dx_0 \\ &= \int_{\mathbb{R}} q(x_0) \left[\int_{\mathbb{R}} \frac{dk}{2\pi} e^{-Dtk^2 + ik(x - x_0)} \right] dx_0. \end{aligned} \quad (38.5)$$

The bracket is the response to unit initial heat concentrated at x_0 . To evaluate it, complete the square, or use

$$\int_{\mathbb{R}} e^{-ak^2 + bk} dk = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}, \quad \Re a > 0. \quad (38.6)$$

Here $a = Dt$ and $b = i(x - x_0)$, so $b^2 = -(x - x_0)^2$. Thus

$$K_D(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right), \quad t > 0, \quad (38.7)$$

and the complete solution of (38.1) is

$$u(x, t) = \int_{\mathbb{R}} K_D(x - x_0, t) q(x_0) dx_0. \quad (38.8)$$

Translation invariance is visible in the dependence on $x - x_0$, rather than on the two positions separately.

38.3 Normalization, initial limit, and smoothing

The change of variable $z = x/\sqrt{4Dt}$ gives

$$\int_{\mathbb{R}} K_D(x, t) dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2} dz = 1.$$

As $t \downarrow 0$, the kernel becomes narrower and higher while retaining unit area. Its precise limiting statement is

$$\begin{aligned} \lim_{t \downarrow 0} \int_{\mathbb{R}} K_D(x, t) \varphi(x) dx &= \lim_{t \downarrow 0} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2} \varphi(\sqrt{4Dt} z) dz \\ &= \varphi(0) \end{aligned} \quad (38.9)$$

for continuous bounded test functions. In distribution notation, $K_D(x, 0) = \delta(x)$. This establishes the initial condition in (38.8) at every continuity point of q .

The moments of the kernel are

$$\int x K_D(x, t) dx = 0, \quad \int x^2 K_D(x, t) dx = 2Dt.$$

The first follows from oddness; the second follows by differentiating a Gaussian integral with respect to its quadratic coefficient. A localized initial distribution therefore spreads over a distance of order $\sqrt{Dt}$. More generally, if q/M is a normalized nonnegative initial profile, with $M = \int q$, its mean remains fixed and its variance obeys

$$\sigma^2(t) = \sigma^2(0) + 2Dt. \quad (38.10)$$

This also follows by multiplying the PDE by x and x^2 , integrating, and integrating by parts twice.

The kernel is positive, and convolution with it produces a smooth function for $t > 0$ under mild assumptions on q . Evolution over two successive time intervals is consistent with evolution over their sum:

$$K_D(\cdot, t) * K_D(\cdot, s) = K_D(\cdot, t + s). \quad (38.11)$$

The Fourier proof is multiplication of $e^{-Dk^2t} e^{-Dk^2s} = e^{-Dk^2(t+s)}$. Recovering an earlier profile would instead multiply by e^{Dk^2t} , greatly amplifying small errors in high-frequency modes. This is why backward diffusion is unstable. On the whole line an integrable localized profile tends to zero at each fixed position as it spreads; a spatially uniform nonzero limiting profile is relevant instead to a finite insulated domain.

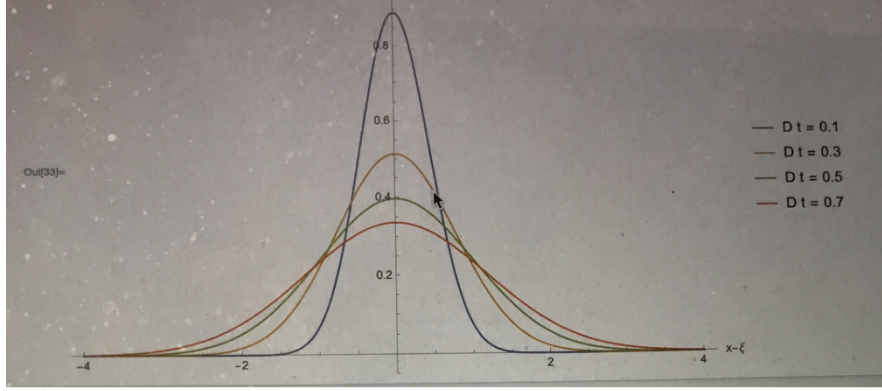


Figure 22: The Gaussian diffusion profile at successive times in the original lecture notes. Its width increases while its area remains fixed.

38.4 The same evolution kernel from a spectral expansion

Suppose first that the problem is posed on a finite interval with self-adjoint boundary conditions for the Laplacian. Let

$$e_n'' = \lambda_n e_n, \quad \langle e_m, e_n \rangle = \delta_{mn}, \quad q_n = \int e_n(x_0)^* q(x_0) dx_0.$$

For $\gamma u_t = u_{xx}$, write $u(x, t) = \sum_n a_n(t) e_n(x)$. Orthogonality gives $\gamma \dot{a}_n = \lambda_n a_n$ and hence

$$u(x, t) = \sum_n q_n e^{\lambda_n t / \gamma} e_n(x).$$

Moving the integral defining q_n outside the sum identifies the kernel

$$K(x, x_0, t) = \sum_n e^{\lambda_n t / \gamma} e_n(x) e_n(x_0)^*. \quad (38.12)$$

Completeness makes $K(x, x_0, 0) = \delta(x - x_0)$, interpreted in the appropriate weak sense. For standard Dirichlet or Neumann conditions, $\lambda_n \leq 0$, so diffusion damps all nonzero modes.

For example, on $0 < x < L$ with zero endpoint temperatures,

$$K_D(x, x_0, t) = \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi x_0}{L} e^{-D(n\pi/L)^2 t}. \quad (38.13)$$

For insulated endpoints, replace the sine eigenfunctions by cosines and include the normalized constant mode $1/\sqrt{L}$. That zero mode leaves $u(x, t) \rightarrow L^{-1} \int_0^L q(x_0) dx_0$ at late times. On an expanding interval the spectral sum approaches the continuous Fourier integral, recovering (38.7) away from boundaries.

38.5 Dimensional analysis and self-similarity

The heat kernel can also be found without first doing a Fourier integral. In $\gamma K_t = K_{xx}$, the parameter has dimensions $[\gamma] = T/L^2$. The unit-area condition requires

$[K] = L^{-1}$. With a point initial source at the origin there is no other length scale, so the dimensionless spatial combination is $x\sqrt{\gamma/t}$. Reflection symmetry allows the form

$$K(x, t) = \sqrt{\frac{\gamma}{t}} g(\xi), \quad \xi = \frac{\gamma x^2}{t}. \quad (38.14)$$

The normalization becomes

$$\int_{-\infty}^{\infty} g(y^2) dy = 1, \quad y = x\sqrt{\gamma/t}. \quad (38.15)$$

Thus profiles at different times have the same shape after a horizontal rescaling by $\sqrt{t}$ and a vertical rescaling by $1/\sqrt{t}$.

There is also a direct symmetry derivation. If u solves the heat equation, then

$$\tilde{u}(x, t) = Au(bx, b^2t)$$

solves it for any constants A and $b > 0$, since both sides acquire the factor Ab^2 . If $u = K$ has a unit delta initial condition, the initial condition of $\tilde{u}$ is

$$A\delta(bx) = \frac{A}{b}\delta(x).$$

It remains a unit source precisely when $A = b$. Uniqueness in the integrable heat-kernel class therefore implies

$$bK(bx, b^2t) = K(x, t), \quad b > 0. \quad (38.16)$$

Fixing a positive reference time in this identity produces $K = t^{-1/2}F(x/\sqrt{t})$, which is the same scaling as (38.14).

Lemma 38.1 (Scaling and power laws). *If a differentiable function satisfies $\lambda f(\lambda^\alpha t) = f(t)$ for all $\lambda > 0$, with $\alpha \neq 0$, then on $t > 0$ it has the form $f(t) = Ct^{-1/\alpha}$.*

Proof. Differentiate with respect to λ :

$$f(\lambda^\alpha t) + \alpha \lambda^\alpha t f'(\lambda^\alpha t) = 0.$$

At $\lambda = 1$, $f + \alpha t f' = 0$. Separating variables gives $f'/f = -1/(\alpha t)$, and integration gives the asserted power law, including the identically zero solution when $C = 0$. $\square$

38.6 Every chain-rule step in the similarity reduction

For the ansatz (38.14),

$$\xi_x = \frac{2\gamma x}{t}, \quad \xi_t = -\frac{\xi}{t}.$$

Consequently,

$$K_t = \sqrt{\gamma} t^{-3/2} \left(-\frac{1}{2}g - \xi g' \right), \quad (38.17)$$

$$K_x = 2 \left(\frac{\gamma}{t} \right)^{3/2} x g',$$

$$K_{xx} = 2 \left(\frac{\gamma}{t} \right)^{3/2} \left(g' + x g'' \frac{2\gamma x}{t} \right) = 2 \left(\frac{\gamma}{t} \right)^{3/2} (g' + 2\xi g''). \quad (38.18)$$

Substitution into $\gamma K_t = K_{xx}$ cancels the common factor and yields

$$4\xi g'' + (\xi + 2)g' + \frac{1}{2}g = 0. \quad (38.19)$$

The useful factorization is

$$\left(\frac{1}{2} + \xi \frac{d}{d\xi}\right) (4g' + g) = 0. \quad (38.20)$$

Writing $\chi = 4g' + g$, one first obtains $\xi\chi' + \chi/2 = 0$, hence $\chi = C\xi^{-1/2}$. For the smooth even heat kernel, g and g' are finite at $\xi = 0$; therefore $C = 0$. The remaining first-order equation is

$$4g' + g = 0, \quad g(\xi) = Ae^{-\xi/4}.$$

Finally, (38.15) gives $1 = A \int e^{-y^2/4} dy = 2A\sqrt{\pi}$, so

$$K(x, t) = \frac{1}{2} \sqrt{\frac{\gamma}{\pi t}} e^{-\gamma x^2/(4t)}.$$

With $\gamma = 1/D$ this is exactly the Fourier result. Regularity at the source and unit normalization, in addition to the PDE, select the physical kernel from the similarity solutions.

38.7 Worked problem: diffusion of an initial Gaussian

Example 38.2 (Green-function reading, Problem 34). Let $q(x) = u_0 e^{-x^2/\ell_0^2}$. Find its subsequent temperature profile.

Insert the initial profile into the convolution:

$$u(x, t) = \frac{u_0}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} \exp\left[-\frac{(x - x_0)^2}{4Dt} - \frac{x_0^2}{\ell_0^2}\right] dx_0.$$

Define $a = 1/(4Dt) + 1/\ell_0^2$ and $b = x/(2Dt)$. The exponent is $-ax_0^2 + bx_0 - x^2/(4Dt)$. Completing the square and applying (38.6) gives

$$\begin{aligned} u(x, t) &= \frac{u_0}{\sqrt{4Dta}} \exp\left(-\frac{x^2}{4Dt} + \frac{b^2}{4a}\right) \\ &= \boxed{\frac{u_0 \ell_0}{\sqrt{\ell_0^2 + 4Dt}} \exp\left[-\frac{x^2}{\ell_0^2 + 4Dt}\right]}. \end{aligned} \quad (38.21)$$

The width parameter satisfies $\ell(t)^2 = \ell_0^2 + 4Dt$, whereas the variance of the normalized density is $\ell(t)^2/2$. The amplitude falls as $\ell_0/\ell(t)$, so

$$\int_{\mathbb{R}} u(x, t) dx = u_0 \ell_0 \sqrt{\pi}$$

is independent of time. Taking $t \rightarrow 0$ recovers the prescribed Gaussian.

38.8 Sources, boundaries, and higher dimensions

For a source $s(x, t)$ switched on after $t = 0$, the problem $u_t - Du_{xx} = s$, $u(x, 0) = q(x)$ has the solution

$$u(x, t) = \int_{\mathbb{R}} K_D(x - x_0, t) q(x_0) dx_0 + \int_0^t d\tau \int_{\mathbb{R}} dx_0 K_D(x - x_0, t - \tau) s(x_0, \tau). \quad (38.22)$$

To check the source term, differentiate the upper limit of the time integral. Since $K_D(x, 0) = \delta(x)$, that term is $s(x, t)$; the remaining integral obeys the homogeneous heat equation. Equivalently, $H(t)K_D(x, t)$ satisfies $(\partial_t - D\partial_x^2)[H(t)K_D] = \delta(t)\delta(x)$.

On the half-line $x > 0$, reflection gives the kernels

$$K_D(x, x_0, t) = K_D(x - x_0, t) - K_D(x + x_0, t),$$

$$K_N(x, x_0, t) = K_D(x - x_0, t) + K_D(x + x_0, t).$$

The first vanishes at $x = 0$; the derivative of the second vanishes there. Both retain the delta initial condition for $x, x_0 > 0$.

In d dimensions the Fourier integral separates into d Gaussian integrals, giving

$$K_D^{(d)}(\mathbf{x}, t) = \frac{e^{-|\mathbf{x}|^2/(4Dt)}}{(4\pi Dt)^{d/2}}, \quad \int_{\mathbb{R}^d} K_D^{(d)} d^d x = 1. \quad (38.23)$$

Each coordinate has variance $2Dt$, so the mean-square distance from a point source is $2dDt$.

39 The free Schrödinger equation and Gaussian wave packets

39.1 Fourier evolution and the propagator

The free one-dimensional Schrödinger equation is

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx}, \quad \psi(x, 0) = \psi_0(x). \quad (39.1)$$

Equivalently, $-i\gamma\psi_t = \psi_{xx}$ with $\gamma = 2m/\hbar > 0$. Its Fourier modes satisfy

$$\partial_t \hat{\psi}(k, t) = -i\frac{\hbar k^2}{2m} \hat{\psi}(k, t), \quad \hat{\psi}(k, t) = \hat{\psi}_0(k) e^{-i\hbar k^2 t/(2m)}.$$

Diffusion suppresses Fourier amplitudes; free quantum evolution changes only their phases. Parseval's identity then shows that $\int |\psi(x, t)|^2 dx$ is conserved.

Example 39.1 (Green-function reading, Problem 35). Derive the free-particle propagator.

Inverting the Fourier transform gives

$$\psi(x, t) = \int_{\mathbb{R}} K_S(x - x_0, t) \psi_0(x_0) dx_0, \quad K_S(x, t) = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikx - i\hbar k^2 t / (2m)}.$$

For $t > 0$, define the oscillatory integral by inserting $e^{-\epsilon k^2}$ and taking $\epsilon \downarrow 0$. Applying (38.6) with $a = \epsilon + i\hbar t / (2m)$, $b = ix$ gives

$$K_S(x, t) = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{imx^2}{2\hbar t}\right) = \frac{e^{-i\pi/4}}{2} \sqrt{\frac{\gamma}{\pi t}} e^{i\gamma x^2 / (4t)}. \quad (39.2)$$

The branch is fixed by continuity from $\Re a > 0$, so $\sqrt{i} = e^{i\pi/4}$. The Fresnel integral

$$\int_{\mathbb{R}} e^{iy^2} dy = \sqrt{\pi} e^{i\pi/4}$$

is understood with the same convergence prescription. It gives $\int K_S dx = 1$ in the oscillatory sense, and the Fourier multiplier tends to one as $t \rightarrow 0$, giving the delta initial condition. Unlike the heat kernel, this kernel is complex and is not a probability density; physical position probabilities are $|\psi|^2$.

39.2 Worked problem: a moving Gaussian packet

Example 39.2 (Green-function reading, Problem 36). The initial state is

$$\psi_0(x) = A \exp\left(-\frac{x^2}{\ell_0^2} + ik_0 x\right).$$

Normalize it, evolve it, and find its velocity and width.

Normalization requires

$$1 = |A|^2 \int_{\mathbb{R}} e^{-2x^2/\ell_0^2} dx = |A|^2 \ell_0 \sqrt{\frac{\pi}{2}}.$$

Choosing an irrelevant overall phase to be zero,

$$A = \left(\frac{2}{\pi \ell_0^2}\right)^{1/4}. \quad (39.3)$$

The convolution with (39.2) contains the exponent

$$\frac{im(x - x_0)^2}{2\hbar t} - \frac{x_0^2}{\ell_0^2} + ik_0 x_0 = -ax_0^2 + bx_0 + \frac{imx^2}{2\hbar t},$$

where

$$a = \frac{1}{\ell_0^2} - \frac{im}{2\hbar t}, \quad b = ik_0 - \frac{imx}{\hbar t}.$$

Since $\Re a > 0$, the ordinary complex Gaussian integral applies. On simplifying its prefactor and exponent, introduce

$$v_0 = \frac{\hbar k_0}{m}, \quad t_0 = \frac{m \ell_0^2}{2\hbar}, \quad s = \frac{t}{t_0}.$$

The result is

$$\psi(x, t) = \frac{A}{\sqrt{1 + is}} \exp \left[-\frac{(x - v_0 t)^2}{\ell_0^2(1 + is)} + ik_0 x - i \frac{\hbar k_0^2 t}{2m} \right]. \quad (39.4)$$

The square root is the branch continuous from 1 at $t = 0$. Taking the absolute square uses $\Re[(1 + is)^{-1}] = (1 + s^2)^{-1}$ and gives

$$|\psi(x, t)|^2 = \frac{|A|^2}{\sqrt{1 + s^2}} \exp \left[-\frac{2(x - v_0 t)^2}{\ell_0^2(1 + s^2)} \right]. \quad (39.5)$$

Thus

$$\langle x \rangle = v_0 t, \quad \ell(t) = \ell_0 \sqrt{1 + (t/t_0)^2}, \quad \Delta x(t) = \frac{\ell(t)}{2}. \quad (39.6)$$

The group velocity obtained from the dispersion relation $\omega(k) = \hbar k^2/(2m)$ is $\omega'(k_0) = \hbar k_0/m = v_0$, consistent with the motion of the center. The momentum distribution is time independent, with $\langle p \rangle = \hbar k_0$ and $\Delta p = \hbar/\ell_0$. Initially $\Delta x \Delta p = \hbar/2$. At late times, $\Delta x(t) \sim (\Delta p/m)t$: components with different momenta separate because they have different velocities.

40 The wave equation and characteristic propagation

40.1 Newton's law for a stretched string

Let $u(x, t)$ be the small transverse displacement of a string with uniform mass per length ρ and tension T . A short element of length δx has mass $\rho \delta x$. The horizontal force balance gives

$$T_1 \cos \theta_1 = T_2 \cos \theta_2.$$

For small slopes, $\cos \theta \simeq 1$, so the tension is approximately constant. The vertical equation of motion is

$$\rho \delta x u_{tt} = T \sin \theta_2 - T \sin \theta_1.$$

Using $\sin \theta \simeq \tan \theta = u_x$, the right side is $T[u_x(x + \delta x, t) - u_x(x, t)]$. Divide by δx and take its limit:

$$\rho u_{tt} = T u_{xx}, \quad u_{tt} - c^2 u_{xx} = 0, \quad c = \sqrt{\frac{T}{\rho}}. \quad (40.1)$$

The approximation neglects terms of second and higher order in the slope in the force equation. The wave speed is determined by the ratio of restoring tension to inertia.

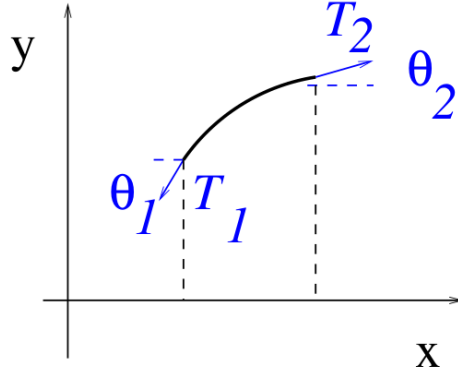


Figure 23: Forces on a short element of a stretched string, reproduced from the wave-equation lecture.

The same equation follows from the action

$$S[u] = \int_{t_1}^{t_2} dt \int_0^L dx \left(\frac{\rho}{2} u_t^2 - \frac{T}{2} u_x^2 \right). \quad (40.2)$$

The first term is kinetic energy per length. The extra length of a displaced element is $[\sqrt{1 + u_x^2} - 1]dx \simeq u_x^2 dx/2$, giving the second term. For variations vanishing on the prescribed boundaries,

$$\delta S = \int dt \int dx (\rho u_t \delta u_t - T u_x \delta u_x) = \int dt \int dx (-\rho u_{tt} + T u_{xx}) \delta u.$$

Stationarity for arbitrary δu gives (40.1). Since it is second order in time, the initial-value problem requires both a displacement $U(x) = u(x, 0)$ and a velocity $V(x) = u_t(x, 0)$.

40.2 Factorization and characteristic coordinates

For constant c , define

$$L_+ = \partial_t + c\partial_x, \quad L_- = \partial_t - c\partial_x.$$

The two operators commute, and

$$\partial_t^2 - c^2 \partial_x^2 = L_+ L_- = L_- L_+.$$

A right-moving profile $f(x - ct)$ satisfies $L_+ f = 0$; a left-moving profile $g(x + ct)$ satisfies $L_- g = 0$. Each is a solution of the wave equation. To establish that their sum gives every sufficiently smooth solution, set

$$\xi = x - ct, \quad \eta = x + ct, \quad x = \frac{\xi + \eta}{2}, \quad t = \frac{\eta - \xi}{2c}.$$

The chain rule gives

$$\partial_x = \partial_\xi + \partial_\eta, \quad \partial_t = -c\partial_\xi + c\partial_\eta,$$

and therefore

$$\partial_t^2 - c^2 \partial_x^2 = -4c^2 \partial_\xi \partial_\eta.$$

The wave equation reduces to $u_{\xi\eta} = 0$. Integrate first with respect to η to get $u_\xi = F(\xi)$, and then with respect to ξ :

$$\boxed{u(x, t) = f(x - ct) + g(x + ct).} \quad (40.3)$$

The arbitrary functions replace the arbitrary constants familiar from ODEs. Their arguments label the two families of characteristics $x - ct = \text{const.}$ and $x + ct = \text{const.}$.

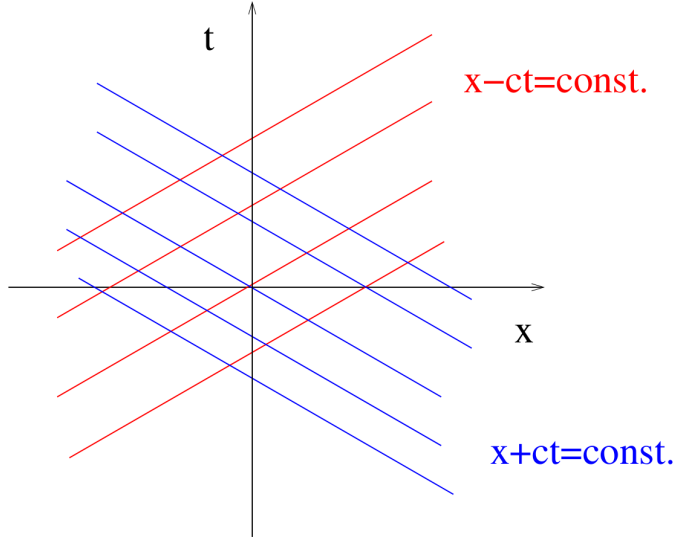


Figure 24: The two families of wave characteristics in the space–time plane.

40.3 Characteristic curves from the principal matrix

For the general second-order equation

$$Au_{tt} + 2Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = R,$$

the principal matrix in coordinates (t, x) is $Q = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$. A level curve $\chi(t, x) = \text{const.}$ is characteristic when

$$(\nabla\chi)^\top Q \nabla\chi = A\chi_t^2 + 2B\chi_t\chi_x + C\chi_x^2 = 0. \quad (40.4)$$

Geometrically, its normal $n = \nabla\chi$ is orthogonal to Qn , or Qn vanishes. Along the curve, $\chi_t + \chi_x x'(t) = 0$. If $\chi_x \neq 0$, substitute $x' = -\chi_t/\chi_x$ into (40.4) to obtain

$$A(x')^2 - 2Bx' + C = 0. \quad (40.5)$$

There are two distinct real characteristic directions when $B^2 - AC > 0$, one repeated direction when $B^2 - AC = 0$, and no real directions when $B^2 - AC < 0$. For $u_{tt} - c^2 u_{xx} = 0$, this gives $x' = \pm c$.

The importance of a noncharacteristic initial curve can be checked without guessing a solution. Suppose u and its normal derivative are specified on $G(t, x) = 0$. A tangent vector is $\tau = (-G_x, G_t)$. Differentiating the prescribed value of u along the curve determines its tangential derivative; together with the normal derivative, this determines u_t and u_x . Their tangential derivatives give two equations for the second derivatives:

$$-G_x u_{tt} + G_t u_{tx} = \tau \cdot \nabla u_t, \quad -G_x u_{tx} + G_t u_{xx} = \tau \cdot \nabla u_x.$$

Together with the PDE these form a 3×3 linear system whose coefficient determinant is

$$\det \begin{pmatrix} A & 2B & C \\ -G_x & G_t & 0 \\ 0 & -G_x & G_t \end{pmatrix} = AG_t^2 + 2BG_tG_x + CG_x^2. \quad (40.6)$$

The second derivatives can be recovered uniquely when this is nonzero. It vanishes exactly on characteristic data curves. For analytic coefficients and analytic data, the noncharacteristic condition is part of the local Cauchy–Kovalevskaya theorem. Existence in that theorem and continuous stability of a Cauchy problem are distinct requirements; the wave equation with data at $t = 0$ has the usual well-posed evolution.

40.4 Derivation of d’Alembert’s initial-value formula

Impose the data $u(x, 0) = U(x)$ and $u_t(x, 0) = V(x)$ on (40.3). They give

$$f(x) + g(x) = U(x), \quad -cf'(x) + cg'(x) = V(x).$$

Integrating the second equation from a fixed reference point b yields

$$g(x) - f(x) = \frac{1}{c} \int_b^x V(s) ds + C.$$

Adding and subtracting this relation and $f + g = U$ gives

$$\begin{aligned} f(x) &= \frac{1}{2}U(x) - \frac{1}{2c} \int_b^x V(s) ds - \frac{C}{2}, \\ g(x) &= \frac{1}{2}U(x) + \frac{1}{2c} \int_b^x V(s) ds + \frac{C}{2}. \end{aligned}$$

The constants cancel when reconstructing u , and the two integrals combine into an integral between the characteristic feet:

$$u(x, t) = \frac{U(x - ct) + U(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} V(s) ds. \quad (40.7)$$

At $t = 0$ the integral vanishes and the average gives $U(x)$. Differentiating with respect to t shows that the two U' terms cancel at $t = 0$, while the endpoint contributions give $V(x)$.

Only data within $[x - ct, x + ct]$ can affect $u(x, t)$. This interval is the **domain of dependence** on the initial line. Conversely, a localized change of data can affect only points reachable at speed c . If $V = 0$, the initial displacement splits into two identical profiles, each with half the initial amplitude, moving in opposite directions.

Example 40.1 (A rectangular displacement released from rest). Take $U(x) = U_0$ for $|x| < a$, $U(x) = 0$ for $|x| > a$, and $V = 0$. Then

$$u(x, t) = \frac{U_0}{2} [\mathbf{1}_{|x-ct|<a} + \mathbf{1}_{|x+ct|<a}]. \quad (40.8)$$

The right-moving pulse occupies $ct - a < x < ct + a$, and the left-moving pulse occupies $-ct - a < x < -ct + a$. Before $t = a/c$, the overlap $ct - a < x < a - ct$ retains height U_0 ; each exposed side has height $U_0/2$. At $t = a/c$ the two pulses meet only at a boundary point, and afterward they separate with an expanding region of zero displacement between them. Values at discontinuity points can be assigned by the limiting half-sum convention. The ideal sharp profile is interpreted weakly.

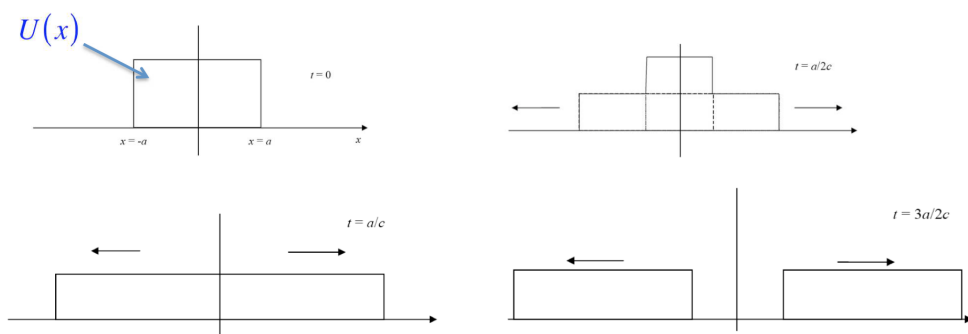


Figure 25: An initial rectangular pulse released from rest separates into two counter-propagating pulses of half amplitude.

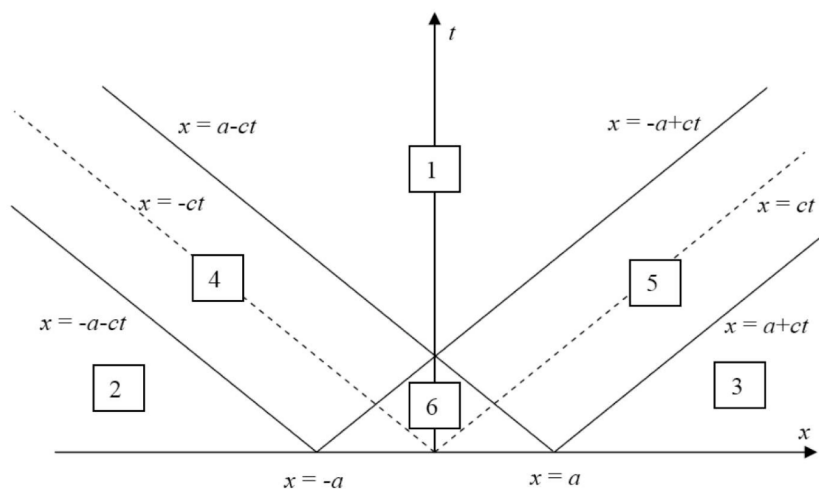


Figure 26: Characteristic boundaries of the rectangular-pulse solution. The central early-time region receives both traveling contributions; the two outward strips receive one each.

40.5 Fourier solution and its equivalence to d'Alembert's formula

Fourier transformation in x gives an oscillator for every wave number:

$$\hat{u}_{tt} + c^2 k^2 \hat{u} = 0.$$

With $\omega_k = c|k|$, its solution is

$$\hat{u}(k, t) = \hat{U}(k) \cos(\omega_k t) + \hat{V}(k) \frac{\sin(\omega_k t)}{\omega_k}. \quad (40.9)$$

At $k = 0$, the last factor is its continuous limit t . The inverse transform of the cosine term follows from $\cos(ckt) = (e^{ickt} + e^{-ickt})/2$, producing the two shifted copies of U . For the velocity term use

$$\frac{\sin(ckt)}{ck} = \frac{1}{2c} \int_{-ct}^{ct} e^{iks} ds.$$

After interchanging the k and s integrals, this gives $(2c)^{-1} \int_{-ct}^{ct} V(x+s) ds$, exactly the second term of (40.7). The characteristic and spectral solutions are two representations of the same evolution.

An equivalent traveling-frequency form is

$$\begin{aligned} \hat{u}(k, t) &= a(k)e^{-i\omega_k t} + b(k)e^{i\omega_k t}, \\ a(k) &= \frac{1}{2} \left(\hat{U}(k) + \frac{i\hat{V}(k)}{\omega_k} \right), \quad b(k) = \frac{1}{2} \left(\hat{U}(k) - \frac{i\hat{V}(k)}{\omega_k} \right). \end{aligned}$$

For a real displacement the coefficients satisfy the corresponding complex-conjugacy conditions between k and $-k$.

40.6 Traveling sinusoids, phases, and standing waves

A right-moving harmonic wave can be written in any of the forms

$$\begin{aligned} u(x, t) &= A \sin(kx - \omega t + \delta) \\ &= A \sin[k(x - ct) + \delta] \\ &= A \sin \left[2\pi \left(\frac{x}{\lambda} - \frac{t}{\mathcal{T}} \right) + \delta \right], \end{aligned}$$

where

$$c = \frac{\omega}{k} = \frac{\lambda}{\mathcal{T}}, \quad \lambda = \frac{2\pi}{k}, \quad \mathcal{T} = \frac{2\pi}{\omega}, \quad \nu = \frac{1}{\mathcal{T}} = \frac{\omega}{2\pi}$$

for $k, \omega > 0$. Reversing both signs of the phase does not reverse the propagation direction: $\sin(\omega t - kx)$ still travels to the right. A complex amplitude encodes phase efficiently: $u = \Re[\mathcal{A}e^{i(kx - \omega t)}]$.

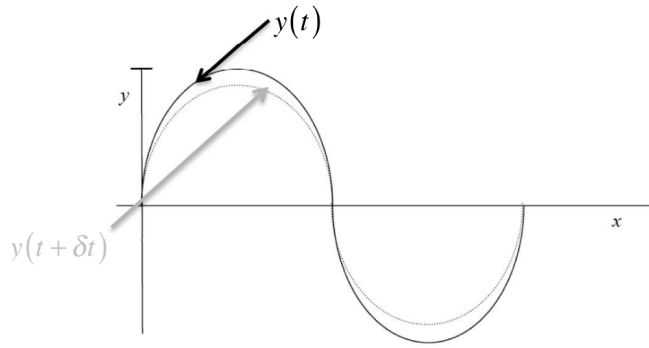
Equal-amplitude waves traveling in opposite directions produce

$$A \sin(kx - \omega t) + A \sin(kx + \omega t) = 2A \sin(kx) \cos(\omega t). \quad (40.10)$$

The profile no longer translates. Every point oscillates with the same frequency, with a position-dependent amplitude. Nodes occur at $kx = n\pi$ and antinodes at $kx = (n + 1/2)\pi$. More generally, the trigonometric sum formula gives

$$\begin{aligned} A \sin(kx - \omega t + 2\delta_1) + A \sin(kx + \omega t + 2\delta_2) \\ = 2A \sin(kx + \delta_1 + \delta_2) \cos(\omega t - \delta_1 + \delta_2). \end{aligned} \quad (40.11)$$

The phases shift the nodes and the origin of time, while the standing-wave character remains unchanged.



“stationary wave”

Figure 27: Snapshots of a standing wave: the nodes remain fixed while its amplitude oscillates.

40.7 Reflections and the role of boundary conditions

For a string occupying $x > 0$, the incoming wave toward the left endpoint is $g(x + ct)$ and the reflected wave is $f(x - ct)$. A fixed endpoint requires $u(0, t) = 0$, hence $f(-ct) = -g(ct)$. Thus the reflection reverses the displacement. A free endpoint requires $u_x(0, t) = 0$, giving $f'(-ct) + g'(ct) = 0$, and the reflected displacement has the same sign. These conditions can be implemented in d'Alembert's formula by extending the initial data oddly across a fixed endpoint, or evenly across a free endpoint.

For a string fixed at both ends, $u(0, t) = u(L, t) = 0$, separation of variables $u = X(x)T_0(t)$ gives

$$\frac{X''}{X} = \frac{\ddot{T}_0}{c^2 T_0} = -k^2.$$

The boundary conditions require $X_n(x) = \sin(n\pi x/L)$, $n = 1, 2, \dots$, with $\omega_n = cn\pi/L$. Expanding the initial data in this basis gives

$$u(x, t) = \sum_{n=1}^{\infty} \left[U_n \cos \omega_n t + \frac{V_n}{\omega_n} \sin \omega_n t \right] \sin \frac{n\pi x}{L}, \quad (40.12)$$

where

$$U_n = \frac{2}{L} \int_0^L U(x) \sin \frac{n\pi x}{L} dx, \quad V_n = \frac{2}{L} \int_0^L V(x) \sin \frac{n\pi x}{L} dx.$$

This is the eigenfunction method referred to in the wave-equation lecture. An odd $2L$ -periodic extension of the data gives an equivalent reflection construction. With two free endpoints, cosine modes replace sine modes and the zero-frequency uniform translation must also be included. With one fixed and one free endpoint, the wave numbers become $k_n = (n + 1/2)\pi/L$, $n \geq 0$.

40.8 Worked example: radiation damping of a bead

A bead of mass M slides vertically without friction on a rigid rod at $x = 0$. It is attached to a very long string extending to $x > 0$, with tension T and mass per length ρ . Let $y(t)$ be the bead's transverse displacement. The coupling imposes

$$y(t) = u(0, t), \quad \dot{y}(t) = u_t(0, t).$$

Assume the string has no incoming waves from infinity. Then the wave produced by the bead travels only toward increasing x :

$$u(x, t) = f(x - ct), \quad c = \sqrt{T/\rho}.$$

At the bead,

$$\dot{y} = -cf'(-ct), \quad u_x(0, t) = f'(-ct) = -\frac{\dot{y}}{c}.$$

The transverse component of the tension is $F_y \simeq Tu_x(0, t)$. Newton's law therefore becomes

$$\boxed{M\ddot{y} + \frac{T}{c}\dot{y} = 0, \quad \frac{T}{c} = \sqrt{T\rho}.} \quad (40.13)$$

For $y(0) = y_0$ and $\dot{y}(0) = v_0$,

$$\dot{y}(t) = v_0 e^{-t/\tau}, \quad y(t) = y_0 + v_0 \tau (1 - e^{-t/\tau}), \quad \tau = \frac{Mc}{T}.$$

There is no restoring spring in this example, so the bead approaches a constant displaced position, rather than oscillating about the origin.

The effective damping does not imply microscopic friction. The string energy density and flux are

$$\mathcal{E} = \frac{\rho}{2} u_t^2 + \frac{T}{2} u_x^2, \quad \mathcal{S} = -T u_t u_x.$$

For the outgoing wave at $x = 0$, $\mathcal{S}(0, t) = (T/c)\dot{y}^2$. Multiplying (40.13) by $\dot{y}$ gives

$$\frac{d}{dt} \left(\frac{1}{2} M \dot{y}^2 \right) = -\frac{T}{c} \dot{y}^2 = -\mathcal{S}(0, t).$$

The bead's lost kinetic energy is exactly the energy carried away by the string. A finite string would eventually return a reflected wave, so the simple local damping equation applies before that return, or for a genuinely outgoing infinite-string problem.

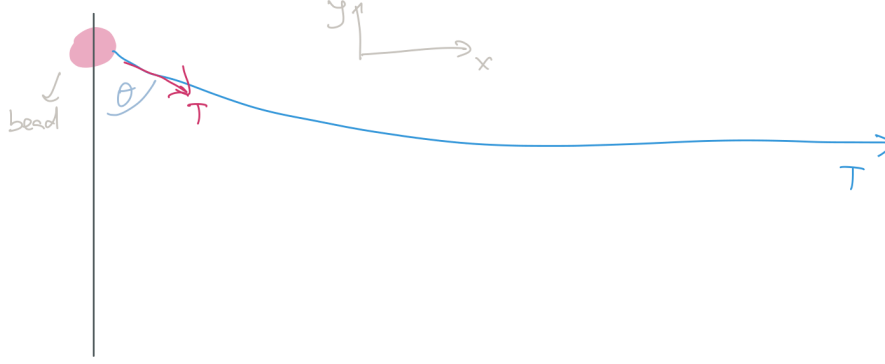


Figure 28: A bead sliding on a frictionless rod is coupled to an outgoing-wave string.

41 Wave Green functions and causal boundary conditions

41.1 A source localized in space and time

To describe wave generation, introduce a source:

$$(\partial_t^2 - c^2 \partial_x^2)u(x, t) = s(x, t). \quad (41.1)$$

The retarded Green function is defined by

$$(\partial_t^2 - c^2 \partial_x^2)G_R(x - x', t - t') = \delta(x - x')\delta(t - t'), \quad G_R = 0 \quad (t < t'). \quad (41.2)$$

A pulse at (x', t') can influence only the future of that event. For zero initial data and sources starting at $t = 0$,

$$u(x, t) = \int_0^t dt' \int_{\mathbb{R}} dx' G_R(x - x', t - t') s(x', t'). \quad (41.3)$$

The differential operator acts on x, t , so applying it under the integral reproduces $s(x, t)$ from the two delta functions.

41.2 The outgoing one-dimensional Helmholtz kernel

It is useful to derive the frequency-domain kernel in the sign convention of the Green-function example reading. Define

$$\mathcal{W} = \partial_x^2 - c^{-2} \partial_t^2, \quad \mathcal{W}\mathcal{G} = \delta(x - x')\delta(t - t').$$

The two operator conventions are related by $\mathcal{W} = -c^{-2}(\partial_t^2 - c^2 \partial_x^2)$, so

$$\mathcal{G} = -c^2 G_R. \quad (41.4)$$

Use the temporal transform $\tilde{\mathcal{G}}(x, x'; \omega) = \int e^{i\omega\tau} \mathcal{G}(x, x'; \tau) d\tau$. Then

$$\left(\frac{d^2}{dx^2} + k^2 \right) \tilde{\mathcal{G}}(x, x'; \omega) = \delta(x - x'), \quad k = \frac{\omega}{c}. \quad (41.5)$$

At real positive ω , outgoing boundary conditions require a left-moving wave for $x < x'$ and a right-moving wave for $x > x'$:

$$\tilde{\mathcal{G}} = \begin{cases} Ae^{-ik(x-x')}, & x < x', \\ Be^{ik(x-x')}, & x > x'. \end{cases}$$

Continuity gives $A = B$. Integrating the differential equation across x' gives a jump in its first derivative,

$$\partial_x \tilde{\mathcal{G}}(x' + 0) - \partial_x \tilde{\mathcal{G}}(x' - 0) = 1.$$

Thus $ikB + ikA = 1$, and

$$\boxed{\tilde{\mathcal{G}}(x, x'; \omega) = \frac{e^{ik|x-x'|}}{2ik}.} \quad (41.6)$$

When $c = 1$, this is $e^{i\omega|x-x'|}/(2i\omega)$. Reflecting boundary conditions on a finite interval would give a different kernel; the differential equation alone does not specify a Green function.

41.3 The pole prescription and contour evaluation

The inverse temporal transform of (41.6) formally contains a pole at $\omega = 0$. Causality selects the prescription

$$\tilde{\mathcal{G}}_R(r; \omega) = \lim_{\epsilon \downarrow 0} \frac{ce^{i(\omega+i\epsilon)r/c}}{2i(\omega+i\epsilon)}, \quad r = |x - x'|. \quad (41.7)$$

The pole is just below the real axis. Equivalently, integrate along the real line with a small detour above the pole. The inverse integral is

$$\mathcal{G}_R(r, \tau) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \frac{d\omega}{2\pi} \frac{ce^{-i\omega(\tau-r/c)}e^{-\epsilon r/c}}{2i(\omega+i\epsilon)}.$$

If $\tau < r/c$, close in the upper half-plane, where the exponential falls off; no pole is enclosed, so the result is zero. If $\tau > r/c$, close in the lower half-plane. The contour is clockwise, so the real-axis integral is $-2\pi i$ times the residue, giving $-c/2$. Therefore

$$\mathcal{G}_R(r, \tau) = -\frac{c}{2}H(\tau - r/c), \quad \boxed{G_R(x, \tau) = \frac{1}{2c}H(\tau - |x|/c).} \quad (41.8)$$

Both vanish before the source occurs and before its wavefront can reach the observation point. The value assigned to $H(0)$ has no effect on these distributional solutions.

41.4 The driven solution and its connection to initial velocity

For $\partial_t^2 - c^2\partial_x^2$, inserting (41.8) gives

$$u_{\text{forced}}(x, t) = \frac{1}{2c} \int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} s(x', t') dx'. \quad (41.9)$$

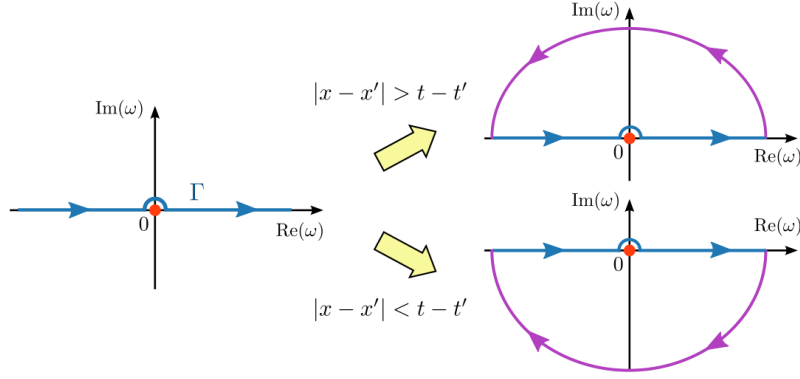


Figure 29: The inverse-transform contour passes above the pole. Closing above or below depends on whether the observation event is outside or inside the future wave cone.

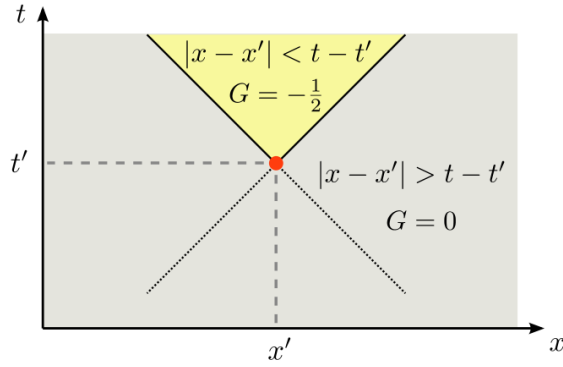


Figure 30: Space-time support of the retarded kernel. This original diagram uses $c = 1$ and the operator $\partial_x^2 - \partial_t^2$, for which the value inside the cone is $\mathcal{G}_R = -1/2$.

The integration region is the past characteristic cone of (x, t) . Add the homogeneous solution (40.7) to include nonzero initial displacement and velocity.

An alternative check comes from integrating the Green equation across $t = 0$. Causality gives $G_R(x, 0^-) = 0$; the jump condition gives

$$G_R(x, 0^+) = 0, \quad \partial_t G_R(x, 0^+) = \delta(x).$$

D'Alembert's formula with $U = 0$ and $V = \delta$ is precisely (41.8). More generally,

$$u_{\text{hom}}(x, t) = \int_{\mathbb{R}} [\partial_t G_R(x - x', t) U(x') + G_R(x - x', t) V(x')] dx'. \quad (41.10)$$

The derivative of the step kernel places the displacement data on the two wavefronts, while the initial velocity contributes throughout the interval between them.

41.5 Three dimensions and geometric spreading

The outgoing three-dimensional Helmholtz kernel solves

$$(\nabla^2 + k^2)\mathcal{G}_\omega(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}').$$

Away from the source, a radial function satisfies $\nabla^2 g = r^{-2}(r^2 g)'$. Writing $g(r) = v(r)/r$ gives $v'' + k^2 v = 0$, so outgoing radiation selects $v = Ae^{ikr}$. Integrate the Green equation over a sphere of radius ϵ around the source. The k^2 volume term tends to zero, and the flux condition is

$$1 = \lim_{\epsilon \rightarrow 0} 4\pi\epsilon^2 g'(\epsilon) = -4\pi A.$$

Thus

$$\mathcal{G}_\omega(\mathbf{r}, \mathbf{r}') = -\frac{e^{ikR}}{4\pi R}, \quad R = |\mathbf{r} - \mathbf{r}'|. \quad (41.11)$$

Inverting in time and using (41.4) yields

$$\boxed{G_R^{(3)}(R, \tau) = \frac{\delta(\tau - R/c)}{4\pi c^2 R}.} \quad (41.12)$$

For a source present in the past, the retarded solution is

$$u(\mathbf{r}, t) = \int_{\mathbb{R}^3} \frac{s(\mathbf{r}', t - R/c)}{4\pi c^2 R} d^3 r'.$$

The source is sampled at its retarded time. In three dimensions the pulse is supported on the wavefront. In one dimension the kernel remains nonzero throughout the interior of the cone. In two dimensions, for comparison, the retarded kernel is

$$G_R^{(2)}(R, t) = \frac{H(t - R/c)}{2\pi c \sqrt{c^2 t^2 - R^2}},$$

which also has an interior tail. The difference is a property of propagation in different spatial dimensions.

The amplitude in (41.11) falls as $1/R$. Its energy intensity therefore falls as $1/R^2$, compensating the growing area $4\pi R^2$ of a sphere. An outgoing one-dimensional wave has no such geometrical dilution because it is confined to a line.

41.6 Nonuniform media and vector Green functions

For a spatially varying refractive index, the scalar frequency-domain problem becomes

$$\left[\nabla^2 + n(\mathbf{r})^2 \frac{\omega^2}{c^2} \right] \mathcal{G}(\mathbf{r}, \mathbf{r}'; \omega) = \delta^{(3)}(\mathbf{r} - \mathbf{r}'). \quad (41.13)$$

Translation invariance is lost: the Green function generally depends on both positions separately, and scattering by the inhomogeneity prevents a purely radial outgoing

profile. Except for special geometries, the boundary-value problem is solved numerically or by approximation.

Electromagnetic waves introduce another distinction: both fields and sources are vectors. The response to a source component J_j can have field components E_i , requiring a matrix-valued or **dyadic** kernel G_{ij} . In vacuum, with time dependence $e^{-i\omega t}$,

$$(\nabla \times \nabla \times - k^2 I) \mathbf{E} = i\omega\mu_0 \mathbf{J}.$$

A dyadic Green function obeys this operator equation with right side $I\delta^{(3)}(\mathbf{r} - \mathbf{r}')$, and the field is obtained by integrating it against the current. The explicit construction and the static inverse-Laplacian limit are developed in the electromagnetic applications. The same inverse-operator idea extends to quantum propagation; quantum field-theory Green functions describe correlations of field operators, with additional ordering and state information.

42 Elliptic equations and capillary surface deformation

42.1 Laplace, Helmholtz, and modified Helmholtz equations

The Laplace equation is $\nabla^2 u = 0$. Its solutions describe source-free static fields and equilibrium temperature profiles. A time-harmonic wave $u(\mathbf{r}, t) = \Re[U(\mathbf{r})e^{-i\omega t}]$ instead satisfies

$$(\nabla^2 + k^2)U = 0, \quad k = \omega/c,$$

the Helmholtz equation. A restoring term with the opposite sign gives

$$(-\nabla^2 + \kappa^2)h = 0, \quad \kappa > 0, \tag{42.1}$$

the modified Helmholtz equation. Its decaying solutions are appropriate to screening and capillary relaxation in space. These problems are elliptic and require spatial boundary data rather than wave-like Cauchy data on a time slice.

42.2 Surface energy, gravitational energy, and capillary length

Consider a liquid surface $z = h(x, y)$ whose flat reference state is $h = 0$. Write its surface tension as σ , its density contrast with the medium above as $\Delta\rho > 0$, and gravitational acceleration as g . For air above a liquid one often approximates $\Delta\rho$ by the liquid density ρ .

The area of a surface element is $\sqrt{1 + h_x^2 + h_y^2} dx dy$. Subtracting the flat area, its excess surface energy is

$$\sigma(\sqrt{1 + |\nabla h|^2} - 1) dx dy.$$

For the gravitational contribution, raise the local column from height 0 to height h . The energy per horizontal area is

$$\int_0^h \Delta\rho g z dz = \frac{1}{2} \Delta\rho g h^2.$$

This expression is also positive for a depression: the energy is measured relative to the hydrostatic flat reference state. The full excess energy is therefore

$$E[h] = \int dx dy \left[\sigma(\sqrt{1 + |\nabla h|^2} - 1) + \frac{1}{2} \Delta \rho g h^2 \right]. \quad (42.2)$$

For $|\nabla h| \ll 1$, use $\sqrt{1 + s} = 1 + s/2 + O(s^2)$:

$$E[h] \simeq \frac{1}{2} \int dx dy [\sigma |\nabla h|^2 + \Delta \rho g h^2]. \quad (42.3)$$

42.3 Variation of the energy

Let $h \mapsto h + \epsilon \eta$, with $\eta = 0$ on boundaries where the height is prescribed. Differentiating (42.3) gives

$$\begin{aligned} \left. \frac{dE[h + \epsilon \eta]}{d\epsilon} \right|_0 &= \int [\sigma \nabla h \cdot \nabla \eta + \Delta \rho g h \eta] dA \\ &= \int [-\sigma \nabla^2 h + \Delta \rho g h] \eta dA + \sigma \int_{\partial \Omega} \eta \partial_n h ds. \end{aligned} \quad (42.4)$$

The boundary term vanishes for fixed height. Since η is arbitrary in the interior, equilibrium requires

$$-\sigma \nabla^2 h + \Delta \rho g h = 0, \quad (-\nabla^2 + \ell_c^{-2})h = 0, \quad \boxed{\ell_c = \sqrt{\frac{\sigma}{\Delta \rho g}}}. \quad (42.5)$$

The capillary length compares surface-tension and gravitational energies: a variation on length ℓ has contributions of order $\sigma h^2/\ell^2$ and $\Delta \rho g h^2$. They balance at $\ell = \ell_c$. The units confirm that ℓ_c is a length, since $[\sigma] = \text{energy/area}$ and $[\Delta \rho g] = \text{energy/volume/length}$.

42.4 A nonstraight wire and a Fourier boundary solution

Suppose a thin wire along $y = 0$ imposes the height $h(x, 0) = f(x)$. Solve on $y > 0$, imposing $h(x, y) \rightarrow 0$ as $y \rightarrow \infty$. Fourier transform only the coordinate parallel to the boundary:

$$h(x, y) = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{h}(k, y) e^{ikx}.$$

Equation (42.5) becomes

$$\hat{h}_{yy} - (k^2 + \ell_c^{-2})\hat{h} = 0. \quad (42.6)$$

Writing $\alpha_k = \sqrt{k^2 + \ell_c^{-2}}$, the general transformed solution is $A_k e^{-\alpha_k y} + B_k e^{\alpha_k y}$. Decay forces $B_k = 0$, and the boundary condition fixes $A_k = \hat{f}(k)$. Hence

$$\boxed{h(x, y) = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{f}(k) e^{ikx - y \sqrt{k^2 + \ell_c^{-2}}}. \quad (42.7)$$

If liquid lies on both sides of the wire, solve the two half-planes separately, replacing y by $|y|$. A derivative jump at the wire records the force it applies, rather than a failure of the field equation away from that boundary.

For a sinusoidal wire $f(x) = a \cos(k_0 x)$,

$$h(x, y) = a \cos(k_0 x) e^{-y \sqrt{k_0^2 + \ell_c^{-2}}}. \quad (42.8)$$

Its penetration depth is $(k_0^2 + \ell_c^{-2})^{-1/2}$. Fine boundary features decay over their own short wavelength scale; broad features penetrate no farther than order ℓ_c .

42.5 Worked problem: the boundary Green kernel

The lecture asks for the kernel in the representation

$$h(x, y) = \int_{\mathbb{R}} P_{\kappa}(x - x', y) f(x') dx', \quad \kappa = \ell_c^{-1}.$$

Equation (42.7) already identifies its Fourier form:

$$P_{\kappa}(x, y) = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ikx - y \sqrt{k^2 + \kappa^2}}, \quad y > 0. \quad (42.9)$$

To express it using a modified Bessel function, introduce

$$F(x, y) = \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{e^{ikx - y \sqrt{k^2 + \kappa^2}}}{2 \sqrt{k^2 + \kappa^2}}.$$

This is the two-dimensional full-space Green function of $-\nabla^2 + \kappa^2$ evaluated away from its source. Rotational symmetry and decay give

$$F(x, y) = \frac{1}{2\pi} K_0(\kappa R), \quad R = \sqrt{x^2 + y^2}. \quad (42.10)$$

The normalization follows from $K_0(z) \sim -\log z$ near zero: the outward flux of $-\nabla F$ through a small circle is one. Differentiating the Fourier expression shows $P_{\kappa} = -2\partial_y F$. Use $K'_0(z) = -K_1(z)$ and $\partial_y R = y/R$ to obtain

$$\boxed{P_{\kappa}(x, y) = \frac{\kappa y}{\pi \sqrt{x^2 + y^2}} K_1\left(\kappa \sqrt{x^2 + y^2}\right)}. \quad (42.11)$$

Its transform tends to 1 as $y \downarrow 0$, so it approaches $\delta(x)$ and supplies the specified boundary data. Its integral over x is

$$\int_{\mathbb{R}} P_{\kappa}(x, y) dx = e^{-\kappa y},$$

consistent with the constant boundary profile $h(x, y) = f_0 e^{-\kappa y}$. This is a boundary propagation kernel, rather than a mass-conserving heat kernel.

In the zero-gravity limit $\kappa \rightarrow 0$, the asymptotic behavior $K_1(z) \sim 1/z$ gives the upper-half-plane Poisson kernel

$$P_0(x, y) = \frac{y}{\pi(x^2 + y^2)}, \quad h(x, y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{yf(x')}{(x - x')^2 + y^2} dx'. \quad (42.12)$$

It also follows directly by integrating $\pi^{-1} \int_0^\infty e^{-ky} \cos(kx) dk$. The modified Helmholtz kernel therefore continuously connects capillary screening to the harmonic extension of boundary data.

Part VIII

Electrostatic and Magnetostatic Applications

Electrostatic
and mag-
netic
fields

43 Electrostatic potentials and the inverse Laplacian

43.1 From Maxwell's equations to Poisson's equation

In electrostatics, the charge distribution and electric field are time independent. Maxwell's equations reduce to

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}, \quad \nabla \times \mathbf{E} = 0.$$

In a simply connected region, the vanishing curl allows a scalar potential Φ with $\mathbf{E} = -\nabla\Phi$. Substitution into Gauss's law gives

$$-\nabla^2\Phi = \frac{\rho_e}{\epsilon_0}. \quad (43.1)$$

Where the charge density is zero, Φ satisfies Laplace's equation. A constant added to the potential leaves the field unchanged; its value is fixed only after a reference or boundary value is specified.

43.2 The fundamental solution and its delta normalization

For $r > 0$, the radial Laplacian gives

$$\nabla^2 \frac{1}{r} = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \frac{1}{r} \right) = 0.$$

This does not describe the origin, where $1/r$ is singular. Integrate the Laplacian over a small ball and use the divergence theorem:

$$\int_{B_\epsilon} \nabla^2 \frac{1}{r} dV = \int_{S_\epsilon} \nabla \frac{1}{r} \cdot \hat{\mathbf{n}} dS = \left(-\frac{1}{\epsilon^2} \right) 4\pi\epsilon^2 = -4\pi.$$

The entire contribution is concentrated at the origin, so

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \quad (43.2)$$

Thus the free-space kernel for $-\nabla^2$ is

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

For a localized source and potential tending to zero at infinity,

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \frac{\rho_e(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'.} \quad (43.3)$$

Applying $-\nabla^2$ under the integral and using (43.2) verifies the equation. Differentiating once gives Coulomb's field formula,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho_e(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'.$$

The Green kernel replaces an arbitrary charge density by a continuous superposition of point-source responses.

43.3 Green's identities and boundary data

The product rule for divergence gives

$$\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \nabla^2 v.$$

After integration over a region V , Green's first identity is

$$\int_V (u \nabla^2 v + \nabla u \cdot \nabla v) dV = \int_{\partial V} u \partial_n v dS. \quad (43.4)$$

Interchange u, v and subtract to get Green's second identity,

$$\int_V (u \nabla^2 v - v \nabla^2 u) dV = \int_{\partial V} (u \partial_n v - v \partial_n u) dS. \quad (43.5)$$

All normal derivatives here use the outward normal to V .

Let $s = \rho_e/\epsilon_0$ and choose a Green function satisfying $-\nabla'^2 G(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r}' - \mathbf{r})$. For fixed observation point $\mathbf{r}$, apply (43.5) in the primed coordinate with $u = \Phi(\mathbf{r}')$, $v = G(\mathbf{r}, \mathbf{r}')$. Its left side is $-\Phi(\mathbf{r}) + \int_V G s dV'$, so

$$\Phi(\mathbf{r}) = \int_V G s dV' + \int_{\partial V} [G \partial_{n'} \Phi - \Phi \partial_{n'} G] dS'. \quad (43.6)$$

The boundary integral contains the information omitted by a free-space source integral. Different Green functions implement different boundary conditions for the same differential operator.

For prescribed potential $\Phi = f$ on ∂V , choose the Dirichlet Green function $G_D = 0$ there. Then

$$\boxed{\Phi(\mathbf{r}) = \int_V G_D(\mathbf{r}, \mathbf{r}') s(\mathbf{r}') dV' - \int_{\partial V} f(\mathbf{r}') \partial_{n'} G_D(\mathbf{r}, \mathbf{r}') dS'.} \quad (43.7)$$

The first term is due to charge in the volume; the second propagates the specified boundary potential into the interior.

43.4 Uniqueness and the Neumann compatibility condition

Suppose Φ_1 and Φ_2 solve the same Poisson equation with the same Dirichlet data. Their difference $w = \Phi_1 - \Phi_2$ satisfies $\nabla^2 w = 0$ and $w = 0$ on the boundary. Use $u = v = w$ in (43.4):

$$\int_V |\nabla w|^2 dV = \int_{\partial V} w \partial_n w dS = 0.$$

Thus $\nabla w = 0$, so w is constant on a connected region; its zero boundary value forces that constant to vanish. This proves uniqueness. The same argument for common Neumann data $\partial_n \Phi = g$ shows only that w is constant, which is the expected potential ambiguity.

Neumann data cannot be assigned arbitrarily. Integrating $-\nabla^2 \Phi = s$ gives

$$\int_{\partial V} g dS = - \int_V s dV. \quad (43.8)$$

The total normal field must agree with the total charge. For a bounded region, a convenient zero-mean Neumann Green function satisfies

$$-\nabla'^2 G_N = \delta^{(3)}(\mathbf{r}' - \mathbf{r}) - \frac{1}{|V|}, \quad \partial_{n'} G_N = 0, \quad \int_V G_N dV' = 0.$$

The subtracted constant removes the zero mode of the Laplacian. Applying Green's identity now gives

$$\Phi(\mathbf{r}) - \bar{\Phi} = \int_V G_N s dV' + \int_{\partial V} G_N g dS', \quad \bar{\Phi} = \frac{1}{|V|} \int_V \Phi dV. \quad (43.9)$$

This is the PDE version of removing a zero eigenvalue before inverting a self-adjoint operator.

43.5 Worked boundary problem: a grounded conducting plane

Let the physical region be $z > 0$, with $\Phi = 0$ on the plane $z = 0$. For a source point $\mathbf{r}' = (x', y', z')$, reflect it to $\mathbf{r}'^* = (x', y', -z')$. The kernel

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'^*|} \right] \quad (43.10)$$

has the required point singularity in $z > 0$. The image singularity lies outside the physical domain. On $z = 0$ the two distances are equal, so $G_D = 0$. Its decay at infinity and the uniqueness theorem complete the verification.

For a point charge q at height a above the origin, cylindrical radius $R = \sqrt{x^2 + y^2}$ gives

$$\Phi(R, z) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{R^2 + (z - a)^2}} - \frac{1}{\sqrt{R^2 + (z + a)^2}} \right]. \quad (43.11)$$

The electric field immediately above the conductor is

$$E_z(R, 0^+) = -\frac{qa}{2\pi\epsilon_0(R^2 + a^2)^{3/2}}.$$

Using the outward normal from the conductor into $z > 0$, the induced surface charge is

$$\sigma_{\text{ind}}(R) = \epsilon_0 E_z(R, 0^+) = -\frac{qa}{2\pi(R^2 + a^2)^{3/2}}.$$

Its total is $2\pi \int_0^\infty R \sigma_{\text{ind}}(R) dR = -q$. The image charge is a mathematical device for enforcing the boundary condition; the physical negative charge resides on the conducting plane.

44 Spherical boundaries and electrostatic multipoles

44.1 Separating Laplace's equation in spherical coordinates

The spherical Laplacian is

$$\nabla^2 = \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2} \Delta_{S^2},$$

where the angular eigenfunctions satisfy $\Delta_{S^2} Y_{\ell m} = -\ell(\ell + 1) Y_{\ell m}$. For a separated harmonic potential $\Phi = R_\ell(r) Y_{\ell m}(\Omega)$,

$$\frac{d}{dr} (r^2 R'_\ell) - \ell(\ell + 1) R_\ell = 0.$$

Try $R_\ell = r^p$. The resulting equation is $p(p + 1) = \ell(\ell + 1)$, whose roots are $p = \ell$ and $p = -\ell - 1$. Hence the general expansion in a charge-free spherical shell is

$$\Phi(r, \Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [a_{\ell m} r^\ell + b_{\ell m} r^{-\ell-1}] Y_{\ell m}(\Omega). \quad (44.1)$$

The radial factor is selected by the location of the physical region: regularity at $r = 0$ excludes $r^{-\ell-1}$, while decay at infinity excludes r^ℓ .

44.2 Prescribed potential on a sphere

Suppose $\Phi(a, \Omega) = f(\Omega)$ on a sphere of radius a . Expand

$$f(\Omega) = \sum_{\ell, m} f_{\ell m} Y_{\ell m}(\Omega), \quad f_{\ell m} = \int_{S^2} Y_{\ell m}(\Omega)^* f(\Omega) d\Omega.$$

For a source-free interior, regularity and matching each harmonic give $a_{\ell m} a^\ell = f_{\ell m}$, so

$$\Phi_{\text{in}}(r, \Omega) = \sum_{\ell, m} f_{\ell m} (r/a)^\ell Y_{\ell m}(\Omega). \quad (44.2)$$

For a source-free exterior with $\Phi \rightarrow 0$ at infinity, $b_{\ell m} a^{-\ell-1} = f_{\ell m}$, giving

$$\boxed{\Phi_{\text{out}}(r, \Omega) = \sum_{\ell, m} f_{\ell m} (a/r)^{\ell+1} Y_{\ell m}(\Omega).} \quad (44.3)$$

Each angular component has its own radial power. This is why orthogonality is so useful: projecting onto the angular basis turns a boundary function into independent algebraic matching conditions.

For axisymmetric data there is no ϕ dependence, so only $m = 0$ contributes. Write $f(\theta) = \sum_{\ell} c_{\ell} P_{\ell}(\cos \theta)$, where

$$c_{\ell} = \frac{2\ell + 1}{2} \int_{-1}^1 f(\arccos \mu) P_{\ell}(\mu) d\mu.$$

For example, $f(\theta) = V_0 \cos \theta$ contains only $\ell = 1$:

$$\Phi_{\text{in}} = V_0(r/a) \cos \theta = V_0 z/a, \quad \Phi_{\text{out}} = V_0(a/r)^2 \cos \theta.$$

The interior field is uniform, $\mathbf{E} = -V_0 \hat{\mathbf{z}}/a$, while the exterior potential has dipolar angular and radial dependence.

44.3 The inverse-distance expansion inside a source integral

The spherical-harmonic addition formula derived in Section 30 gives

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell + 1} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} Y_{\ell m}(\Omega) Y_{\ell m}(\Omega')^*, \quad (44.4)$$

with $r_{<} = \min(r, r')$ and $r_{>} = \max(r, r')$. If the observation radius lies outside all sources, $r > r'$, insert this expansion into (43.3) and define

$$Q_{\ell m} = \int_{\mathbb{R}^3} r'^{\ell} Y_{\ell m}(\Omega')^* \rho_e(\mathbf{r}') d^3 r'. \quad (44.5)$$

Then

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{\ell, m} \frac{Q_{\ell m}}{2\ell + 1} \frac{Y_{\ell m}(\Omega)}{r^{\ell+1}}.} \quad (44.6)$$

The ℓ th multipole falls as $r^{-\ell-1}$ in potential. At distances large compared with the source size, the lowest nonzero multipole therefore dominates.

44.4 Monopole, dipole, and quadrupole terms

Let $\hat{\mathbf{r}} = \mathbf{r}/r$. Expanding the inverse distance in powers of r'/r gives

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r^2} \\ &+ \frac{3(\hat{\mathbf{r}} \cdot \mathbf{r}')^2 - r'^2}{2r^3} + O(r'^3/r^4). \end{aligned} \quad (44.7)$$

This follows either from the Legendre generating function or by applying $(1+s)^{-1/2} = 1 - s/2 + 3s^2/8 + \dots$ to $s = -2\hat{\mathbf{r}} \cdot \mathbf{r}'/r + r'^2/r^2$. Define the total charge, dipole moment, and traceless quadrupole tensor:

$$Q = \int \rho_e(\mathbf{r}') d^3r', \quad (44.8)$$

$$p_i = \int r'_i \rho_e(\mathbf{r}') d^3r', \quad (44.9)$$

$$Q_{ij} = \int (3r'_i r'_j - r'^2 \delta_{ij}) \rho_e(\mathbf{r}') d^3r'. \quad (44.10)$$

Then the potential is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{2r^3} \sum_{i,j} Q_{ij} \hat{r}_i \hat{r}_j + O(r^{-4}) \right]. \quad (44.11)$$

If $Q = 0$, a nonzero dipole is the leading far field. If the distribution is also inversion symmetric, its dipole vanishes and the quadrupole or a higher moment supplies the leading term. Symmetry can therefore remove many coefficients before any integration is performed.

44.5 The shell theorem from angular orthogonality

Integrating (44.4) over the source direction kills all $\ell > 0$ terms, because only the constant spherical harmonic has nonzero angular average. Thus

$$\int_{S^2} \frac{d\Omega'}{|\mathbf{r} - r'\hat{\mathbf{r}}'|} = \frac{4\pi}{r_>}. \quad (44.12)$$

For a shell of radius a carrying uniform total charge Q , its surface charge density is $Q/(4\pi a^2)$. The potential becomes

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0} \begin{cases} a^{-1}, & r < a, \\ r^{-1}, & r > a. \end{cases} \quad (44.13)$$

The electric field is zero inside and equals the field of a point charge Q at the origin outside. The constant interior potential is required by continuity at the shell; zero field does not require zero potential.

For a general spherical density, the same angular integral yields

$$\Phi(r) = \frac{1}{\epsilon_0} \left[\frac{1}{r} \int_0^r r'^2 \rho_e(r') dr' + \int_r^\infty r' \rho_e(r') dr' \right]. \quad (44.14)$$

On differentiating, the endpoint terms cancel, leaving

$$E_r(r) = \frac{1}{\epsilon_0 r^2} \int_0^r r'^2 \rho_e(r') dr' = \frac{Q_{\text{enc}}(r)}{4\pi\epsilon_0 r^2}.$$

The spherical-harmonic calculation thereby reproduces Gauss's-law result.

45 Magnetostatics and vector potentials

45.1 Current conservation and gauge freedom

For steady currents, charge conservation gives $\nabla \cdot \mathbf{J} = 0$. The magnetostatic equations are

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Represent the divergence-free magnetic field as $\mathbf{B} = \nabla \times \mathbf{A}$. The vector potential is not unique:

$$\mathbf{A} \mapsto \mathbf{A} + \nabla \chi$$

leaves $\mathbf{B}$ unchanged because $\nabla \times \nabla \chi = 0$. Choose the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. The identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

then transforms Ampère's law into

$$-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \quad (45.1)$$

In Cartesian coordinates this is one scalar Poisson equation for each component. With suitable decay at infinity,

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'.} \quad (45.2)$$

For localized divergence-free current, this expression indeed satisfies Coulomb gauge. Since $\nabla R^{-1} = -\nabla' R^{-1}$,

$$\begin{aligned} \nabla \cdot \mathbf{A} &= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \cdot \nabla' R^{-1} d^3 r' \\ &= \frac{\mu_0}{4\pi} \int R^{-1} \nabla' \cdot \mathbf{J} d^3 r' = 0, \end{aligned}$$

where the surface term vanishes for localized currents.

45.2 Deriving the Biot–Savart law

Take the curl of (45.2). The primed current does not depend on the observation coordinate, so

$$\nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{R} \right) = \nabla(R^{-1}) \times \mathbf{J}(\mathbf{r}') = \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{R^3}.$$

Therefore

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r'.} \quad (45.3)$$

For a thin wire carrying constant current I , the current-volume element becomes $\mathbf{J} d^3 r' = I d\ell'$, giving

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\ell' \times (\mathbf{r} - \mathbf{r}')}{R^3}.$$

The scalar Coulomb kernel is therefore the common mathematical source of both electrostatic and magnetostatic free-space solutions.

45.3 The leading field of a localized current distribution

Expanding the inverse distance in (45.2), the candidate $1/r$ term is proportional to $\int \mathbf{J} d^3r'$. For a localized steady current this integral vanishes. Componentwise,

$$\partial'_j(r'_i J_j) = J_i + r'_i \partial'_j J_j = J_i,$$

so its volume integral is a vanishing surface term. The next term is

$$A_i(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r^2} \hat{r}_j \int r'_j J_i(\mathbf{r}') d^3r'.$$

The symmetric part of the integral vanishes because integrating $\partial'_k(r'_i r'_j J_k)$ gives $\int (r'_i J_j + r'_j J_i) d^3r' = 0$. The remaining antisymmetric part is described by the magnetic dipole moment

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}') d^3r'. \quad (45.4)$$

Consequently,

$$\mathbf{A}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}, \quad \mathbf{B}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r^3} [3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}] \quad (45.5)$$

away from the source region. For a planar current loop, $\mathbf{m} = I\mathbf{S}$, where the oriented area vector follows the right-hand rule. This parallels the electrostatic multipole expansion, with the absence of a magnetic monopole reflected in the leading dipole field.

46 Electromagnetic waves, polarization, and dyadic response

46.1 Deriving the vacuum wave equations

In a source-free vacuum, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad (46.1)$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \partial_t \mathbf{E}. \quad (46.2)$$

Take the curl of Faraday's law:

$$\nabla \times \nabla \times \mathbf{E} = -\partial_t (\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \partial_t^2 \mathbf{E}.$$

Using $\nabla \cdot \mathbf{E} = 0$, the curl-curl identity makes the left side $-\nabla^2 \mathbf{E}$. Thus

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \partial_t^2 \mathbf{E} = 0, \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}. \quad (46.3)$$

The same steps applied to Ampère's law give

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \partial_t^2 \mathbf{B} = 0. \quad (46.4)$$

Each component satisfies a scalar wave equation, but the components are linked by the divergence and curl equations. Choosing unrelated scalar wave solutions for all six field components would generally violate Maxwell's equations.

46.2 Plane waves and transversality

Write a monochromatic plane wave in complex form:

$$\mathbf{E}(\mathbf{r}, t) = \Re[\mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}], \quad \mathbf{B}(\mathbf{r}, t) = \Re[\mathbf{B}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}].$$

The divergence equations give

$$\mathbf{k} \cdot \mathbf{E}_0 = 0, \quad \mathbf{k} \cdot \mathbf{B}_0 = 0.$$

The fields are transverse to the propagation direction. Faraday's law and Ampère's law become

$$\mathbf{k} \times \mathbf{E}_0 = \omega \mathbf{B}_0, \quad \mathbf{k} \times \mathbf{B}_0 = -\frac{\omega}{c^2} \mathbf{E}_0.$$

Taking $\mathbf{k} \times$ the first relation and using transversality gives $-k^2 \mathbf{E}_0 = -(\omega^2/c^2) \mathbf{E}_0$, hence $\omega = ck$ for a positive-frequency wave. In particular,

$$\boxed{\mathbf{B}_0 = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E}_0.} \quad (46.5)$$

The real fields are mutually perpendicular and have $|\mathbf{B}| = |\mathbf{E}|/c$ for a plane wave. Their Poynting vector $\mathbf{S} = \mu_0^{-1} \mathbf{E} \times \mathbf{B}$ points along propagation.

46.3 Linear, circular, and elliptical polarization

Take propagation along $+z$, and set

$$E_x = A \cos \theta, \quad E_y = B \cos(\theta + \delta), \quad \theta = kz - \omega t, \quad (46.6)$$

with $A, B \geq 0$. At a fixed point, these two components describe the motion of the electric-field tip in the transverse plane. For $A, B > 0$, put $X = E_x/A$ and $Y = E_y/B$. Then

$$Y = X \cos \delta - \sin \theta \sin \delta.$$

Squaring after isolating $\sin \theta$ and using $\sin^2 \theta = 1 - X^2$ eliminates the time variable:

$$\boxed{\frac{E_x^2}{A^2} + \frac{E_y^2}{B^2} - \frac{2E_x E_y}{AB} \cos \delta = \sin^2 \delta.} \quad (46.7)$$

For generic phase difference this is an ellipse. The special cases are

Amplitudes	Phase difference	Polarization
$A > 0, B = 0$	any	linear along x
$A = 0, B > 0$	any	linear along y
$A, B > 0$	0	linear, positive slope B/A
$A, B > 0$	π	linear, negative slope $-B/A$
$A = B > 0$	$+\pi/2$	circular
$A = B > 0$	$-\pi/2$	circular, opposite rotation
$A \neq B$, both nonzero	$\pm\pi/2$	elliptical, axes along x, y

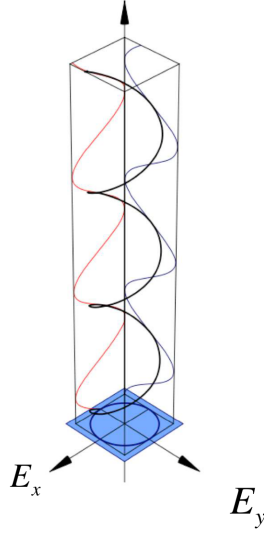


Figure 31: The original lecture illustration of a transverse polarized wave. The transverse electric vector changes direction while the phase propagates along the longitudinal axis.

To specify rotation without a naming ambiguity, choose $z = 0$, so $\theta = -\omega t$. For $\delta = +\pi/2$, $(E_x, E_y) = (A \cos \omega t, B \sin \omega t)$: the field turns from $+x$ toward $+y$ as time increases. The phase $-\pi/2$ reverses that rotation. A label such as left or right circular polarization must also specify the convention and viewing direction.

A complex two-component amplitude collects amplitude and phase:

$$\mathbf{E}_0 = A\hat{x} + Be^{i\delta}\hat{y}.$$

Multiplying $\mathbf{E}_0$ by a common phase shifts the origin of time but does not change the polarization ellipse. Relative phase controls the polarization state.

46.4 Constructing the vacuum dyadic Green function

For a time-harmonic current with convention $e^{-i\omega t}$, Maxwell's curl equations imply

$$(\nabla \times \nabla \times - k^2 I) \mathbf{E} = i\omega\mu_0 \mathbf{J}, \quad k = \omega/c. \quad (46.8)$$

The dyadic kernel $\mathbf{G}$ is defined by

$$(\nabla \times \nabla \times - k^2 I) \mathbf{G}(\mathbf{r}, \mathbf{r}'; \omega) = I \delta^{(3)}(\mathbf{r} - \mathbf{r}').$$

Let

$$g(\mathbf{r}, \mathbf{r}') = \frac{e^{ikR}}{4\pi R}, \quad (\nabla^2 + k^2)g = -\delta^{(3)}(\mathbf{r} - \mathbf{r}').$$

Then for $k \neq 0$ the outgoing dyadic is

$$\boxed{\mathbf{G} = \left(I + \frac{\nabla \nabla}{k^2} \right) g,} \quad (46.9)$$

with derivatives in the observation coordinate and the singularity interpreted distributionally. To verify this, use $\nabla \times \nabla \times = \nabla \nabla \cdot - I \nabla^2$. Applying the wave operator to gI gives $\nabla \nabla g - I(\nabla^2 + k^2)g$. Its action on $k^{-2} \nabla \nabla g$ is $-\nabla \nabla g$, because the curl of a gradient vanishes. The two derivative terms cancel, leaving $I\delta^{(3)}$ as required.

The field generated by a current is consequently

$$E_i(\mathbf{r}, \omega) = i\omega\mu_0 \sum_j \int G_{ij}(\mathbf{r}, \mathbf{r}'; \omega) J_j(\mathbf{r}', \omega) d^3r'. \quad (46.10)$$

The two indices specify the response field component and source-current component. In nonuniform or bounded media, the same operator definition holds with the appropriate material parameters and boundary conditions, but the simple free-space formula must be replaced by the corresponding boundary-value solution.

46.5 Connections among the course methods

Electrostatic and magnetostatic equations use the inverse Laplacian; wave propagation uses a frequency-dependent inverse with an outgoing condition. Spherical harmonics diagonalize angular dependence, Fourier transforms diagonalize translation-invariant derivatives, and Green functions reconstruct the response to general sources. Boundary conditions, regularity, and causality select which inverse is physically relevant. These applications combine the function-space, operator, complex-analysis, and PDE tools developed in the preceding parts.

Part IX

Groups and Representations in Physics

Group
theory

47 Groups, homomorphisms, and matrix examples

A symmetry is a transformation that preserves the structure of a physical problem. Two rotations can be performed successively, every rotation can be undone, and doing nothing is itself an allowed transformation. These observations lead to the definition of a group. The group describes how the transformations compose. A representation will describe how they act on a particular vector space of states or fields.

47.1 Group axioms and their consequences

Definition 47.1. A **group** is a set G equipped with a multiplication $G \times G \rightarrow G$, denoted $(g, h) \mapsto gh$, such that:

1. Multiplication is associative: $(gh)k = g(hk)$.
2. There is an identity e satisfying $eg = ge = g$ for every g .
3. Every g has an inverse g^{-1} satisfying $g^{-1}g = gg^{-1} = e$.

The multiplication is closed in G . The group is **Abelian** if $gh = hg$ for every pair of elements. For a finite group, its **order** is the number $|G|$ of its elements.

Proposition 47.2. *The identity and the inverse of each element are unique. Moreover,*

$$(g^{-1})^{-1} = g, \quad (gh)^{-1} = h^{-1}g^{-1}.$$

Proof. If e and $\tilde{e}$ are both identities, then $e = e\tilde{e} = \tilde{e}$. If a and b are both inverses of g , then

$$a = ae = a(gb) = (ag)b = eb = b.$$

Since g is an inverse of g^{-1} , uniqueness gives the first formula. For the second, multiply in the indicated order:

$$(gh)(h^{-1}g^{-1}) = g(hh^{-1})g^{-1} = e,$$

and likewise $(h^{-1}g^{-1})(gh) = e$. Thus $h^{-1}g^{-1}$ is the inverse of gh . $\square$

The inverse reverses the order because the last operation performed must be undone first. Multiplication by an inverse also gives the cancellation laws: $ag = ah$ implies $g = h$, and $ga = ha$ implies $g = h$. These statements do not require commutativity.

47.2 Subgroups

Definition 47.3. A subset $H \subset G$ is a **subgroup** when it is itself a group under the same multiplication.

A practical test is that H contains e and is closed under multiplication and inverses. Associativity is inherited from G . Equivalently, a nonempty subset is a subgroup if

$$h_1, h_2 \in H \implies h_1 h_2^{-1} \in H.$$

To prove this shorter criterion, choose $h \in H$. Then $hh^{-1} = e \in H$. Taking $h_1 = e$ gives $h_2^{-1} \in H$, and applying the criterion to h_1 and h_2^{-1} gives $h_1 h_2 \in H$.

47.3 Homomorphisms, kernels, and images

Definition 47.4. A **homomorphism** $f : G \rightarrow \tilde{G}$ preserves multiplication:

$$f(gh) = f(g)f(h).$$

Its image and kernel are

$$\text{Im } f = \{f(g) : g \in G\}, \quad \ker f = \{g \in G : f(g) = \tilde{e}\}.$$

A bijective homomorphism is an **isomorphism**; the two groups then have the same multiplication structure.

The following properties explain why the kernel measures the failure of a representation to distinguish group elements.

Proposition 47.5. For a homomorphism $f : G \rightarrow \tilde{G}$,

1. $f(e) = \tilde{e}$ and $f(g^{-1}) = f(g)^{-1}$;
2. $\text{Im } f$ and $\ker f$ are subgroups;
3. f is injective exactly when $\ker f = \{e\}$;
4. f is surjective exactly when $\text{Im } f = \tilde{G}$.

Proof. From $f(e) = f(e)^2$, multiply by $f(e)^{-1}$ to obtain the first identity. Then

$$f(g)f(g^{-1}) = f(gg^{-1}) = \tilde{e},$$

so $f(g^{-1}) = f(g)^{-1}$. Products and inverses of elements of the image remain in the image. If $g, h \in \ker f$, then $f(gh^{-1}) = \tilde{e}$, so the subgroup criterion applies to the kernel as well.

If f is injective, $f(g) = f(e)$ forces $g = e$. Conversely, if the kernel is trivial and $f(g) = f(h)$, then

$$f(h^{-1}g) = f(h)^{-1}f(g) = \tilde{e},$$

so $h^{-1}g = e$ and $g = h$. The statement about surjectivity is its definition. $\square$

The kernel is also a **normal subgroup**: if $k \in \ker f$ and $g \in G$, then

$$f(gkg^{-1}) = f(g)\tilde{e}f(g)^{-1} = \tilde{e}.$$

Consequently, identifying elements that differ by a kernel element gives the quotient relation $G/\ker f \cong \text{Im } f$. We will use this relation for the map from $\text{SU}(2)$ to $\text{SO}(3)$.

47.4 Number systems, cyclic groups, and invertible matrices

For a field $\mathbb{F}$, addition makes $(\mathbb{F}, +)$ an Abelian group, and multiplication makes $(\mathbb{F} \setminus \{0\}, \cdot)$ an Abelian group. The additive identity is 0, whereas the multiplicative identity is 1. A vector space is an Abelian group under vector addition. The integers form the additive group $(\mathbb{Z}, +)$, although the nonzero integers do not form a multiplicative group: for example, 2^{-1} is not an integer.

For an integer $n \geq 1$, the cyclic group is

$$\mathbb{Z}_n = \{0, 1, \dots, n-1\}, \quad k \cdot \ell = (k + \ell) \bmod n.$$

Its identity is 0, the inverse of k is $-k \bmod n$, and every element is obtained by repeatedly adding 1. For example, in $\mathbb{Z}_4$, $3 + 2 = 1$ and $3^{-1} = 1$ in this additive notation.

If V is a finite-dimensional vector space, then

$$\text{GL}(V) = \{A : V \rightarrow V \mid A \text{ is linear and invertible}\}$$

is a group under composition. In a chosen basis this becomes $\text{GL}(n, \mathbb{R})$ or $\text{GL}(n, \mathbb{C})$. Closure follows from $(AB)^{-1} = B^{-1}A^{-1}$. Singular matrices are excluded because they cannot undo their action on vectors.

47.5 Unitary and special unitary groups

The unitary and special unitary groups are

$$U(n) = \{U \in \text{GL}(n, \mathbb{C}) : U^\dagger U = I_n\}, \quad \text{SU}(n) = \{U \in U(n) : \det U = 1\}.$$

A unitary matrix preserves the standard inner product: $(Uv)^\dagger(Uw) = v^\dagger w$. It has inverse $U^\dagger$. For two unitary matrices,

$$(UV)^\dagger(UV) = V^\dagger U^\dagger UV = I_n,$$

so their product is unitary. The determinant-one condition is preserved by products and inverses, making $\text{SU}(n)$ a subgroup of $U(n)$.

For $n = 1$, unitarity says $|z|^2 = 1$, so

$$U(1) = \{e^{i\theta} : \theta \in \mathbb{R}\}.$$

Multiplication adds angles modulo 2π . The map

$$f : \mathbb{Z}_n \longrightarrow U(1), \quad f(k) = e^{2\pi i k/n}$$

is a homomorphism, since $f(k + \ell) = f(k)f(\ell)$. Its kernel consists only of $0 \bmod n$, so it embeds $\mathbb{Z}_n$ as the n th roots of unity.

For $\text{SU}(2)$, let the first column of U be $(\alpha, \beta)^\top$. Normalization gives $|\alpha|^2 + |\beta|^2 = 1$. A normalized orthogonal second column must be a phase multiple of $(-\beta^*, \alpha^*)^\top$. Its phase is precisely the determinant, so imposing $\det U = 1$ gives

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (47.1)$$

Writing α and β in real and imaginary parts identifies $\text{SU}(2)$ with the unit three-sphere in $\mathbb{R}^4$. Thus it has three real parameters and is connected and compact.

47.6 Orthogonal groups and rotations

The real counterparts are

$$O(n) = \{A \in \text{GL}(n, \mathbb{R}) : A^T A = I_n\}, \quad \text{SO}(n) = \{A \in O(n) : \det A = 1\}.$$

Orthogonal matrices preserve Euclidean lengths and scalar products. Taking determinants gives $(\det A)^2 = 1$, so $\det A = \pm 1$. The determinant-one matrices preserve orientation. For example, $\text{diag}(-1, 1, 1)$ is an orthogonal reflection, but is not in $\text{SO}(3)$.

In two dimensions, a unit first column can be written $(\cos \theta, \sin \theta)^T$. Orthogonality and positive determinant then fix the second column, giving

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

The angle-addition identities give $R(\theta)R(\varphi) = R(\theta + \varphi)$, so $e^{i\theta} \mapsto R(\theta)$ is an isomorphism $U(1) \cong \text{SO}(2)$. We use active, right-handed rotations throughout this part. Reversing the sign of the angle gives the equivalent passive coordinate convention.

In three dimensions the matrices in $\text{SO}(3)$ are ordinary spatial rotations. Rotations about different axes generally do not commute. Their noncommutativity will appear in the angular-momentum commutation relations.

48 Representations and irreducible building blocks

48.1 Definition and elementary representations

Definition 48.1. A **linear representation** of G on V is a homomorphism

$$D : G \longrightarrow \text{GL}(V), \quad D(gh) = D(g)D(h).$$

Its dimension is $\dim V$. It is **faithful** when D is injective, or equivalently when $\ker D = \{e\}$.

The identity and inverse are automatically represented by $D(e) = I_V$ and $D(g^{-1}) = D(g)^{-1}$. A representation assigns a linear operation to every group element, while keeping the composition law. It need not assign different matrices to different elements.

The **trivial representation** has a one-dimensional space and $D(g) = 1$ for all g . The **defining representation** of a matrix group is its action by the matrices themselves on column vectors. For $\mathbb{Z}_n$ and $U(1)$, examples of one-dimensional representations are

$$D_q(k) = e^{2\pi i q k / n}, \quad D_q(e^{i\theta}) = e^{i q \theta},$$

where $q \in \{0, \dots, n-1\}$ in the first formula and $q \in \mathbb{Z}$ in the second. Both are unitary because their values have unit modulus.

For example, D_2 of $\mathbb{Z}_4$ maps 0, 2 to 1 and 1, 3 to -1 . It is a valid representation, but is not faithful because its kernel is $\{0, 2\}$.

48.2 Direct sums and tensor products

If D acts on V and $\tilde{D}$ acts on W , their direct sum acts on a pair $(v, w) \in V \oplus W$:

$$(D \oplus \tilde{D})(g) = \begin{pmatrix} D(g) & 0 \\ 0 & \tilde{D}(g) \end{pmatrix}.$$

Block multiplication proves the representation property, and $\dim(V \oplus W) = \dim V + \dim W$.

The tensor product acts on products of vectors by

$$(D \otimes \tilde{D})(g)(v \otimes w) = D(g)v \otimes \tilde{D}(g)w.$$

If $\{e_a\}$ and $\{f_b\}$ are bases, then the $e_a \otimes f_b$ form a basis of $V \otimes W$. Thus its dimension is $(\dim V)(\dim W)$. For matrices, the tensor product is the Kronecker product; for example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes B = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix}.$$

Acting on a product vector shows that

$$(A \otimes B)(C \otimes E)(v \otimes w) = ACv \otimes BEw,$$

so $(A \otimes B)(C \otimes E) = AC \otimes BE$. It follows immediately that $D(g) \otimes \tilde{D}(g)$ is a representation.

A direct sum describes alternative invariant sectors. A tensor product describes two sets of degrees of freedom present together, as for two spins. A tensor product of irreducible representations need not be irreducible.

48.3 Equivalent and unitary representations

Representations D on V and $\tilde{D}$ on $\tilde{V}$ are **equivalent** if an invertible map $P : V \rightarrow \tilde{V}$ obeys

$$\tilde{D}(g) = PD(g)P^{-1} \quad \text{for every } g.$$

The same P must work for all elements. This is a change of basis, not a new symmetry action. The relation can also be written $PD(g) = \tilde{D}(g)P$; such a map is called an **intertwiner**.

On a complex inner-product space, D is **unitary** if

$$\langle D(g)v, D(g)w \rangle = \langle v, w \rangle, \quad D(g)^\dagger D(g) = I.$$

Unitary transformations preserve norms and therefore preserve the normalization of quantum states. The matrix representations of Lorentz transformations on finite-component fields, discussed below, need not be unitary with respect to a positive-definite component inner product.

48.4 Invariant subspaces and complete reducibility

A subspace $W \subset V$ is **invariant** if $D(g)W \subset W$ for all g . Since g^{-1} also belongs to the group, this implies $D(g)W = W$. A representation is **irreducible** if its only invariant subspaces are $\{0\}$ and V . Otherwise it is reducible.

Choose a basis of an invariant W and extend it to a basis of V . The matrices then have the form

$$D(g) = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix}.$$

The lower-left block vanishes because vectors in W stay in W . Reducibility alone does not force the upper-right block to vanish. A representation is **completely reducible** if it is a direct sum of irreducible invariant subspaces, so a single basis makes every matrix block diagonal.

Theorem 48.2. *Every finite-dimensional unitary representation is completely reducible. Every finite-dimensional complex representation of a finite group is equivalent to a unitary representation and hence is completely reducible.*

Proof. First suppose D is unitary and W is invariant. For $v \in W^\perp$ and $w \in W$,

$$\langle D(g)v, w \rangle = \langle v, D(g)^{-1}w \rangle = 0,$$

because $D(g)^{-1}w \in W$. Thus $W^\perp$ is also invariant and $V = W \oplus W^\perp$. If either summand is reducible, repeat the same argument there. Finite dimension guarantees termination in irreducible summands.

For a finite group, start with any positive-definite inner product $\langle \cdot, \cdot \rangle_0$ and define

$$\langle v, w \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle D(g)v, D(g)w \rangle_0.$$

This is positive definite: the identity term is positive for $v = w \neq 0$. For any $h \in G$,

$$\langle D(h)v, D(h)w \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle D(gh)v, D(gh)w \rangle_0 = \langle v, w \rangle_G,$$

because $g \mapsto gh$ permutes the group. An orthonormal basis for this averaged inner product makes all matrices unitary, and the first part of the proof applies. $\square$

For compact groups, the corresponding construction uses the normalized invariant group measure instead of a finite sum:

$$\langle v, w \rangle_G = \int_G \langle D(g)v, D(g)w \rangle_0 d\mu(g).$$

Consequently, finite-dimensional continuous representations of $SU(2)$ can be made unitary. In this case the invariant measure can be viewed as the normalized uniform measure on the three-sphere (47.1).

Example 48.3 (Reducible does not always mean completely reducible). The additive group $(\mathbb{R}, +)$ has a representation

$$D(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad D(t+s) = D(t)D(s).$$

The line spanned by $(1, 0)^\top$ is invariant. Any other line has a vector $(a, b)^\top$ with $b \neq 0$, and $D(t)$ changes its first component by tb . It cannot remain the same line for every t . There is therefore no invariant complementary line, and the representation is not completely reducible.

48.5 Schur's lemma

Lemma 48.4 (Schur's lemma). *Let D and $\tilde{D}$ be finite-dimensional irreducible complex representations. A nonzero intertwiner $P : V \rightarrow \tilde{V}$ is an isomorphism. In particular, if $P : V \rightarrow V$ commutes with every $D(g)$, then $P = \lambda I_V$ for some $\lambda \in \mathbb{C}$.*

Proof. If $v \in \ker P$, then $PD(g)v = \tilde{D}(g)Pv = 0$, so $\ker P$ is invariant. Likewise, if $w = Pv$ lies in $\text{Im } P$, then $\tilde{D}(g)w = PD(g)v$ remains in the image. Irreducibility and $P \neq 0$ force $\ker P = \{0\}$ and $\text{Im } P = \tilde{V}$. Thus P is invertible.

For an endomorphism commuting with D , complex finite dimension ensures that P has an eigenvalue λ . The nonzero eigenspace

$$E_\lambda = \{v : Pv = \lambda v\}$$

is invariant, since $PD(g)v = D(g)Pv = \lambda D(g)v$. Irreducibility gives $E_\lambda = V$, which is precisely $P = \lambda I_V$. $\square$

The complex-field assumption matters: the eigenvalue argument is not valid for every real matrix. Also, commuting with one group element is insufficient; the operator must commute with the entire representation.

Corollary 48.5. *Every finite-dimensional complex irreducible representation of an Abelian group is one-dimensional.*

Proof. For fixed h , commutativity gives $D(h)D(g) = D(g)D(h)$ for every g . Schur's lemma implies $D(h) = \lambda(h)I$. Since h was arbitrary, all matrices are scalar. If $\dim V > 1$, every line is then invariant, contradicting irreducibility. Thus $\dim V = 1$. $\square$

48.6 All irreducible representations of the cyclic group

Because $\mathbb{Z}_n$ is Abelian, its complex irreducible representations are one-dimensional. Such a representation is determined by $\zeta = D(1) \neq 0$, since $D(k) = \zeta^k$. The relation $n \cdot 1 = 0$ in $\mathbb{Z}_n$ requires

$$1 = D(0) = D(n \cdot 1) = \zeta^n.$$

Thus

$$D_q(k) = e^{2\pi i qk/n}, \quad q = 0, 1, \dots, n-1, \quad (48.1)$$

is the complete list. Different values of q give inequivalent one-dimensional representations because a scalar change of basis cannot change a scalar matrix.

These functions also form the discrete Fourier basis. Their orthogonality follows directly from a geometric series:

$$\frac{1}{n} \sum_{k=0}^{n-1} D_q(k)^* D_{q'}(k) = \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i(q'-q)k/n} = \delta_{qq'}.$$

For $q \neq q'$, the numerator $1 - e^{2\pi i(q'-q)}$ vanishes while the geometric-series denominator is nonzero. For $q = q'$, every summand is 1. Consequently a function on $\mathbb{Z}_n$ can be expanded as

$$f(k) = \sum_{q=0}^{n-1} c_q D_q(k), \quad c_q = \frac{1}{n} \sum_{k=0}^{n-1} D_q(k)^* f(k).$$

48.7 Continuous irreducible representations of the circle

Let $d(\theta) = D(e^{i\theta})$ for a continuous one-dimensional representation of $U(1)$. It obeys

$$d(\theta + \varphi) = d(\theta)d(\varphi), \quad d(0) = 1, \quad d(\theta + 2\pi) = d(\theta).$$

A continuous homomorphism of this Lie group is smooth. Differentiating with respect to φ at zero gives $d'(\theta) = d'(0)d(\theta)$. Hence $d(\theta) = e^{a\theta}$ for a constant a . Periodicity requires $e^{2\pi a} = 1$, so $a = iq$ with $q \in \mathbb{Z}$. Therefore

$$D_q(e^{i\theta}) = e^{iq\theta}, \quad q \in \mathbb{Z}. \quad (48.2)$$

The integer label is a charge in the chosen normalization of the $U(1)$ generator. Its integrality follows from the circle's global periodicity; the local one-dimensional Lie algebra alone allows any real generator eigenvalue.

The familiar Fourier orthogonality relation is now

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-iq\theta} e^{iq'\theta} d\theta = \delta_{qq'}.$$

Thus Fourier series decompose functions into the irreducible modes of a circle symmetry.

49 Lie groups, generators, and Lie algebras

49.1 Local coordinates and infinitesimal generators

A matrix Lie group is a matrix group that is also a smooth manifold, with smooth multiplication and inversion. Near the identity, choose real coordinates $t_1, \dots, t_d$ with $g(0) = I$ and independent tangent matrices

$$X_a = \left. \frac{\partial g}{\partial t_a} \right|_{t=0}.$$

Then

$$g(t) = I + \sum_{a=1}^d t_a X_a + O(|t|^2).$$

The real span of these tangent matrices is the Lie algebra $\mathfrak{g}$. Its dimension is the number of independent continuous parameters. For example, $U(1)$ and $SO(2)$ have dimension one, whereas $SU(2)$ and $SO(3)$ have dimension three.

In mathematical notation, the bracket is the matrix commutator $[X, Y] = XY - YX$. In physics it is useful to write

$$X_a = -iT_a, \quad g(t) = I - it_a T_a + O(|t|^2).$$

For unitary groups the T_a are Hermitian. If $[X_a, X_b] = f_{ab}^c X_c$ with real structure constants, then

$$[T_a, T_b] = i f_{ab}^c T_c. \quad (49.1)$$

The factor i in this equation is part of the physics convention. The real tangent space spanned by the matrices X_a is closed under the ordinary matrix commutator.

49.2 Why the commutator belongs to the algebra

Take two one-parameter group elements e^{sX} and e^{tY} . Their small group commutator expands to

$$e^{sX} e^{tY} e^{-sX} e^{-tY} = I + st[X, Y] + O(s^2 t, st^2).$$

This displays $[X, Y]$ as an infinitesimal group direction. Another way to see the same closure is that conjugation preserves the group, and hence preserves its tangent space:

$$e^{sX} Y e^{-sX} = Y + s[X, Y] + O(s^2).$$

Differentiating at $s = 0$ therefore gives another tangent vector.

The commutator is bilinear and antisymmetric. It also satisfies the **Jacobi identity**,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

For example, the first term expands as $XYZ - XZY - YZX + ZYX$; the two cyclic terms cancel these four products pairwise. The Jacobi identity is the algebraic consistency condition obeyed by the structure constants.

49.3 The matrix exponential and the local/global distinction

For a square matrix X , define

$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n X^n}{n!}.$$

The series converges for all finite t , and termwise differentiation gives

$$\frac{d}{dt} e^{tX} = X e^{tX}, \quad e^{0X} = I.$$

Thus exponentiation integrates an infinitesimal generator into a one-parameter subgroup. Because powers of the same X commute, $e^{sX}e^{tX} = e^{(s+t)X}$.

For $X \in \mathfrak{g}$, e^{tX} stays in the identity component of the group. The exponential map covers a neighborhood of the identity, and finite products of exponentials generate the connected identity component. A single exponential need not cover every element of an arbitrary connected matrix Lie group.

For $U(n)$ the spectral theorem does give a logarithm for every element: if

$$U = V \operatorname{diag}(e^{-i\theta_1}, \dots, e^{-i\theta_n}) V^\dagger,$$

then $U = e^{-iH}$ with $H = V \operatorname{diag}(\theta_1, \dots, \theta_n) V^\dagger$ Hermitian. If $U \in \operatorname{SU}(n)$, its eigenphases have sum $2\pi k$. Subtracting $2\pi k$ from one phase does not change U and makes $\operatorname{tr} H = 0$. Thus every $\operatorname{SU}(n)$ element is an exponential of a traceless Hermitian generator.

Local commutation relations do not determine global topology. In particular, $\operatorname{SU}(2)$ and $\operatorname{SO}(3)$ have the same local angular-momentum algebra but different allowed single-valued group representations.

49.4 Generators of the classical matrix groups

For $U = I - i\epsilon T + O(\epsilon^2)$, expand unitarity:

$$U^\dagger U = (I + i\epsilon T^\dagger)(I - i\epsilon T) = I + i\epsilon(T^\dagger - T) + O(\epsilon^2).$$

Therefore $T^\dagger = T$. A Hermitian $n \times n$ matrix has n real diagonal entries and two real parameters for each of the $n(n-1)/2$ off-diagonal pairs, giving

$$\dim U(n) = n + 2 \frac{n(n-1)}{2} = n^2.$$

For $\operatorname{SU}(n)$, the additional condition is

$$\det(I - i\epsilon T) = 1 - i\epsilon \operatorname{tr} T + O(\epsilon^2) = 1,$$

so $\operatorname{tr} T = 0$ and $\dim \operatorname{SU}(n) = n^2 - 1$.

For an orthogonal matrix $R = I - i\epsilon T + O(\epsilon^2)$, reality requires T to be purely imaginary. Orthogonality gives

$$R^\top R = (I - i\epsilon T^\top)(I - i\epsilon T) = I - i\epsilon(T^\top + T) + O(\epsilon^2),$$

so $T^\top = -T$. Such a matrix is automatically Hermitian. There are $n(n-1)/2$ independent entries, and

$$\mathfrak{o}(n) = \mathfrak{so}(n), \quad \dim O(n) = \dim \operatorname{SO}(n) = \frac{n(n-1)}{2}.$$

The disconnected reflection component of $O(n)$ contributes no new infinitesimal directions near the identity.

For $U(1)$ one may use $T = 1$, so the algebra is one-dimensional and Abelian. For the two-dimensional rotation matrix,

$$R(\theta) = e^{-i\theta T}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Since $T^2 = I$, its exponential is $I \cos \theta - iT \sin \theta$, which reproduces $R(\theta)$ explicitly.

49.5 Pauli matrices and the algebra of rotations

The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Each is Hermitian and traceless. Direct multiplication gives $\sigma_i^2 = I_2$, $\sigma_1\sigma_2 = i\sigma_3$, and the cyclic versions, while reversing two distinct factors changes the sign. Together these statements are

$$\sigma_i\sigma_j = \delta_{ij}I_2 + i\epsilon_{ijk}\sigma_k. \quad (49.2)$$

Consequently,

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}I_2, \quad \text{tr}(\sigma_i\sigma_j) = 2\delta_{ij}.$$

The matrices $\tau_i = \sigma_i/2$ form a basis of traceless Hermitian 2×2 matrices, with

$$[\tau_i, \tau_j] = i\epsilon_{ijk}\tau_k.$$

For a unit vector $\hat{n}$, equation (49.2) gives $(\hat{n} \cdot \sigma)^2 = I_2$: the antisymmetric term vanishes when contracted with n_in_j . Therefore

$$U(\hat{n}, \theta) = e^{-i\theta\hat{n} \cdot \sigma/2} = I_2 \cos \frac{\theta}{2} - i\hat{n} \cdot \sigma \sin \frac{\theta}{2}. \quad (49.3)$$

This is an explicit three-parameter form for $\text{SU}(2)$.

For spatial vectors the rotation generators are instead 3×3 :

$$(J_i^{\text{vec}})_{jk} = -i\epsilon_{ijk}.$$

Written out,

$$J_1^{\text{vec}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_2^{\text{vec}} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

$$J_3^{\text{vec}} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Matrix multiplication gives $[J_i^{\text{vec}}, J_j^{\text{vec}}] = i\epsilon_{ijk}J_k^{\text{vec}}$. Thus two-dimensional spinors and three-dimensional vectors use matrices of different sizes that obey the same algebra.

49.6 Differentiating and exponentiating representations

A Lie algebra representation is a linear map $r : \mathfrak{g} \rightarrow \text{End}(V)$ preserving brackets:

$$r([X, Y]) = [r(X), r(Y)].$$

If a group representation D is differentiable, define its generator by

$$D(e^{tX}) = I + t r(X) + O(t^2).$$

Differentiating the group-commutator identity shows that r preserves commutators. In the physics convention, matrices J_a representing T_a therefore satisfy the same relations

$$[J_a, J_b] = i f_{ab}^c J_c.$$

Conversely, the generators determine $D(e^{-itT}) = e^{-itJ}$ locally. A Lie algebra representation integrates to a representation of the connected simply connected covering group. To descend to a quotient group, any covering-group element identified with the identity must act trivially. The integer-versus-half-integer spin distinction below is an explicit example of this global requirement.

50 The double cover and the classification of spin

50.1 Constructing the map from $SU(2)$ to $SO(3)$

Associate a real vector $\mathbf{x}$ with a traceless Hermitian matrix

$$X = \mathbf{x} \cdot \boldsymbol{\sigma} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$

This is a one-to-one real-linear correspondence, with $x_i = \frac{1}{2} \text{tr}(\sigma_i X)$. The Pauli trace identity gives

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2} \text{tr}(XY).$$

For $U \in SU(2)$, the matrix $UXU^\dagger$ is again traceless and Hermitian, so it defines a real linear transformation $R(U)$ by

$$U(\mathbf{x} \cdot \boldsymbol{\sigma})U^\dagger = (R(U)\mathbf{x}) \cdot \boldsymbol{\sigma}.$$

In components,

$$R(U)_{ij} = \frac{1}{2} \text{tr}(\sigma_i U \sigma_j U^\dagger). \quad (50.1)$$

Trace cyclicity proves preservation of scalar products:

$$\begin{aligned} R(U)\mathbf{x} \cdot R(U)\mathbf{y} &= \frac{1}{2} \text{tr}(UXU^\dagger UYU^\dagger) \\ &= \frac{1}{2} \text{tr}(UXYU^\dagger) = \frac{1}{2} \text{tr}(XY) = \mathbf{x} \cdot \mathbf{y}. \end{aligned}$$

Hence $R(U) \in O(3)$. Since $SU(2)$ is connected, $\det R(U)$ cannot jump between $+1$ and -1 ; its value at $U = I$ is $+1$. Thus $R(U) \in SO(3)$.

Composition follows directly from conjugation:

$$(UV)X(UV)^\dagger = U(VXV^\dagger)U^\dagger, \quad R(UV) = R(U)R(V).$$

So R is a group homomorphism.

50.2 Axis-angle rotations, the kernel, and the two-to-one map

Substituting (49.3) and using the Pauli product identity gives

$$R(U)\mathbf{x} = \mathbf{x} \cos \theta + (\hat{n} \times \mathbf{x}) \sin \theta + \hat{n}(\hat{n} \cdot \mathbf{x})(1 - \cos \theta). \quad (50.2)$$

For example, the term linear in the sine comes from

$$i \cos \frac{\theta}{2} \sin \frac{\theta}{2} [\mathbf{x} \cdot \boldsymbol{\sigma}, \hat{n} \cdot \boldsymbol{\sigma}] = \sin \theta (\hat{n} \times \mathbf{x}) \cdot \boldsymbol{\sigma}.$$

The remaining quadratic term uses $(\hat{n} \cdot \boldsymbol{\sigma})(\mathbf{x} \cdot \boldsymbol{\sigma})(\hat{n} \cdot \boldsymbol{\sigma}) = [2\hat{n}(\hat{n} \cdot \mathbf{x}) - \mathbf{x}] \cdot \boldsymbol{\sigma}$. Equation (50.2) is Rodrigues' rotation formula. Every spatial rotation has an axis and an angle, so the map is surjective.

If $R(U) = I_3$, then U commutes with all three Pauli matrices. Commuting with σ_3 makes U diagonal, and commuting with σ_1 makes its two diagonal entries equal. Thus $U = \lambda I_2$. Unitarity and $\det U = 1$ imply $\lambda = \pm 1$, giving

$$\ker R = \{I_2, -I_2\}, \quad \text{SU}(2)/\{\pm I_2\} \cong \text{SO}(3).$$

Each rotation has two preimages, U and $-U$. In particular,

$$U(\hat{n}, 2\pi) = -I_2, \quad U(\hat{n}, 4\pi) = I_2.$$

A spinor changes sign under a 2π rotation, while a vector returns to itself. The sign change does not occur in the corresponding three-dimensional rotation matrix.

50.3 Ladder operators and the Casimir

In a finite-dimensional unitary representation, choose Hermitian J_1, J_2, J_3 satisfying

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$

We set $\hbar = 1$ for these dimensionless generators. Physical angular momenta are $\hbar J_i$. Define

$$J_{\pm} = J_1 \pm iJ_2, \quad J^2 = J_1^2 + J_2^2 + J_3^2.$$

Using $[J_3, J_1] = iJ_2$ and $[J_3, J_2] = -iJ_1$ gives

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_3, \quad J_+^{\dagger} = J_-.$$

Multiplication also gives the useful identities

$$J_- J_+ = J^2 - J_3^2 - J_3, \quad (50.3)$$

$$J_+ J_- = J^2 - J_3^2 + J_3. \quad (50.4)$$

For the Casimir,

$$[J^2, J_a] = \sum_i (J_i [J_i, J_a] + [J_i, J_a] J_i) = 0,$$

since the result contracts the antisymmetric ϵ_{iak} with the symmetric combination $J_i J_k + J_k J_i$. Thus J^2 commutes with every generator and, after exponentiation, with every group element. Schur's lemma implies that $J^2 = cI$ on an irreducible space.

50.4 Highest and lowest weights

Since J_3 is Hermitian, it has a real eigenvalue m and a normalized eigenvector $|m\rangle$. The commutator with $J_\pm$ gives

$$J_3(J_\pm|m\rangle) = (m \pm 1)J_\pm|m\rangle.$$

Whenever the resulting vector is nonzero, it lies in the eigenspace whose eigenvalue differs by one. Finite dimension implies that there are a largest eigenvalue j and a smallest eigenvalue $m_{\min}$. Their ladder endpoints satisfy

$$J_+|j\rangle = 0, \quad J_-|m_{\min}\rangle = 0.$$

Taking expectation values of (50.3) at the highest state gives

$$0 = \|J_+|j\rangle\|^2 = c - j(j+1), \quad c = j(j+1).$$

At the lowest state, equation (50.4) gives $c = m_{\min}(m_{\min} - 1)$. Subtracting,

$$0 = j(j+1) - m_{\min}(m_{\min} - 1) = (j + m_{\min})(j - m_{\min} + 1).$$

The second factor is positive because $j \geq m_{\min}$, so $m_{\min} = -j$. In particular $j \geq 0$.

Successive lowering produces the eigenvalues $j, j-1, \dots, -j$. Their number is $2j+1$, so $2j$ must be a nonnegative integer. The span of this ladder is invariant under J_3, J_+, J_- and hence under the group. Irreducibility forces this span to be the entire representation. There cannot be an additional independent ladder in the same irreducible representation.

50.5 Normalized matrix elements and existence of every spin

For normalized states $|j, m\rangle$, equations (50.3) and (50.4) determine the norms of the ladder vectors. Choosing phases so the coefficients are nonnegative gives

$$J_3|j, m\rangle = m|j, m\rangle, \tag{50.5}$$

$$J_+|j, m\rangle = \sqrt{(j-m)(j+m+1)}|j, m+1\rangle, \tag{50.6}$$

$$J_-|j, m\rangle = \sqrt{(j+m)(j-m+1)}|j, m-1\rangle. \tag{50.7}$$

The coefficients vanish at the appropriate endpoints and are positive between them.

These formulas also construct a representation for each $j = 0, \frac{1}{2}, 1, \dots$. Define $J_3, J_\pm$ by their action on a $(2j+1)$ -dimensional orthonormal basis and put $J_1 = (J_+ + J_-)/2$, $J_2 = (J_+ - J_-)/(2i)$. For example,

$$\begin{aligned} [J_+, J_-]|j, m\rangle &= (j+m)(j-m+1)|j, m\rangle \\ &\quad - (j-m)(j+m+1)|j, m\rangle \\ &= 2m|j, m\rangle, \end{aligned}$$

which verifies $[J_+, J_-] = 2J_3$; the remaining relations follow from the unit change of m . The matrices exponentiate to $\text{SU}(2)$.

To see irreducibility, suppose a nonzero invariant subspace contains a vector with some nonzero m component. A polynomial in J_3 can project onto that component because all m eigenvalues are distinct. Repeated raising and lowering then produces every basis state, so the subspace must be the full space.

Theorem 50.1. *The finite-dimensional continuous complex irreducible representations of $SU(2)$ are labeled by*

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$$

The spin- j representation has dimension $2j+1$, Casimir $J^2 = j(j+1)I$, and generator matrix elements given by (50.5)–(50.7).

Under a 2π rotation around the third axis,

$$e^{-2\pi i J_3} |j, m\rangle = e^{-2\pi i m} |j, m\rangle = (-1)^{2j} |j, m\rangle.$$

Thus the representation descends to an ordinary representation of $SO(3)$ exactly when j is an integer. Half-integer spins represent its double cover $SU(2)$.

50.6 Low-spin matrices

For $j = 0$ there is one basis vector, and all three generators vanish. This is the scalar representation.

For $j = \frac{1}{2}$, use the ordered basis $|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle$. The ladder formulas give

$$J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Therefore $J_i = \sigma_i/2$ and $J^2 = \frac{3}{4}I_2$.

For $j = 1$, use $|1, 1\rangle, |1, 0\rangle, |1, -1\rangle$. Then

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_- = J_+^\dagger,$$

and hence

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

Here $J^2 = 2I_3$. In Cartesian vector notation, the spherical basis

$$\mathbf{e}_{+1} = -\frac{\mathbf{e}_x + i\mathbf{e}_y}{\sqrt{2}}, \quad \mathbf{e}_0 = \mathbf{e}_z, \quad \mathbf{e}_{-1} = \frac{\mathbf{e}_x - i\mathbf{e}_y}{\sqrt{2}}$$

converts the vector generators J_i^{vec} to these matrices. Thus spin one is equivalent to the ordinary vector representation.

51 Orbital angular momentum and addition of spins

51.1 Rotations acting on functions

Let $f(\mathbf{x})$ be a smooth complex-valued scalar function on $\mathbb{R}^3$. A spatial rotation acts by the pullback

$$(D(R)f)(\mathbf{x}) = f(R^{-1}\mathbf{x}).$$

The inverse is essential for the representation law:

$$\begin{aligned} (D(R_1)D(R_2)f)(\mathbf{x}) &= f(R_2^{-1}R_1^{-1}\mathbf{x}) \\ &= f((R_1R_2)^{-1}\mathbf{x}) = (D(R_1R_2)f)(\mathbf{x}). \end{aligned}$$

For a small active rotation, $R\mathbf{x} = \mathbf{x} + \delta\theta \hat{n} \times \mathbf{x} + O(\delta\theta^2)$. Therefore

$$\begin{aligned} f(R^{-1}\mathbf{x}) &= f(\mathbf{x}) - \delta\theta(\hat{n} \times \mathbf{x}) \cdot \nabla f + O(\delta\theta^2) \\ &= (1 - i\delta\theta \hat{n} \cdot \hat{\mathbf{L}})f(\mathbf{x}) + O(\delta\theta^2), \end{aligned}$$

where

$$\hat{\mathbf{L}} = -i\mathbf{x} \times \nabla, \quad \hat{L}_i = -i\epsilon_{ijk}x_j\partial_k. \quad (51.1)$$

These are dimensionless orbital angular-momentum operators.

For example, $\hat{L}_1 = -i(y\partial_z - z\partial_y)$ and $\hat{L}_2 = -i(z\partial_x - x\partial_z)$. Applying their commutator to a smooth test function and using $[x_a\partial_b, x_c\partial_d] = \delta_{bc}x_a\partial_d - \delta_{da}x_c\partial_b$ gives

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k.$$

The second-derivative terms cancel, leaving the same algebra as for finite-dimensional spin matrices.

51.2 Spherical-coordinate expressions and spherical harmonics

Using the chain rule for $x = r \sin \theta \cos \varphi$, $y = r \sin \theta \sin \varphi$, $z = r \cos \theta$, the operators become

$$\begin{aligned} \hat{L}_1 &= i(\sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi), \\ \hat{L}_2 &= i(-\cos \varphi \partial_\theta + \cot \theta \sin \varphi \partial_\varphi), \\ \hat{L}_3 &= -i\partial_\varphi. \end{aligned}$$

The radial derivatives cancel because rotations preserve r . Adding and subtracting the first two operators gives

$$\hat{L}_\pm = e^{\pm i\varphi}(\pm\partial_\theta + i\cot\theta\partial_\varphi).$$

Squaring and summing, with derivatives also acting on the trigonometric coefficients, gives

$$\hat{L}^2 = - \left[\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\varphi^2 \right] = -\Delta_{S^2}. \quad (51.2)$$

Thus the angular part of the three-dimensional Laplacian is the rotation Casimir:

$$\nabla^2 = \frac{1}{r^2} \partial_r (r^2 \partial_r) - \frac{\hat{L}^2}{r^2}.$$

The normalized spherical harmonics satisfy

$$\hat{L}^2 Y_{\ell m} = \ell(\ell + 1) Y_{\ell m}, \quad \hat{L}_3 Y_{\ell m} = m Y_{\ell m}.$$

For each fixed ℓ , the space

$$V_\ell = \text{span}\{Y_{\ell, -\ell}, \dots, Y_{\ell, \ell}\}$$

is invariant under all rotations. The ladder formulas give

$$\hat{L}_\pm Y_{\ell m} = \sqrt{(\ell \mp m)(\ell \pm m + 1)} Y_{\ell, m \pm 1}.$$

Since the ladder connects all $2\ell + 1$ states, this is an irreducible spin- ℓ representation.

Single-valuedness in φ requires $Y(\theta, \varphi + 2\pi) = Y(\theta, \varphi)$, so m is integral. Consequently only integral ℓ occur for scalar functions on the sphere. Spinor fields require a different transformation law and are not ordinary single-valued scalar functions on S^2 .

51.3 Total generators on a tensor product

For two representations,

$$\begin{aligned} D_1(e^{-i\theta T_i}) \otimes D_2(e^{-i\theta T_i}) &= (I - i\theta J_i^{(1)} + \dots) \otimes (I - i\theta J_i^{(2)} + \dots) \\ &= I - i\theta \left(J_i^{(1)} \otimes I + I \otimes J_i^{(2)} \right) + \dots \end{aligned}$$

Hence the total generator is

$$J_i = J_i^{(1)} \otimes I + I \otimes J_i^{(2)}. \quad (51.3)$$

Operators on different factors commute, so

$$[J_i, J_j] = [J_i^{(1)}, J_j^{(1)}] \otimes I + I \otimes [J_i^{(2)}, J_j^{(2)}] = i\epsilon_{ijk} J_k.$$

In the product basis $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$,

$$J_3 |j_1 m_1; j_2 m_2\rangle = (m_1 + m_2) |j_1 m_1; j_2 m_2\rangle.$$

However, this basis usually does not diagonalize total J^2 , because

$$J^2 = (J^{(1)})^2 + (J^{(2)})^2 + 2 \sum_i J_i^{(1)} \otimes J_i^{(2)}.$$

The cross term mixes states having the same total m .

51.4 Clebsch–Gordan decomposition and its proof

Theorem 51.1 (Addition of two spins). *For $j_1, j_2 \in \{0, \frac{1}{2}, 1, \dots\}$,*

$$V_{j_1} \otimes V_{j_2} \cong \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V_j, \quad (51.4)$$

where j increases in integer steps and each summand appears once.

Proof. Assume $j_1 \geq j_2$. The unique state with largest total weight is $|j_1, j_1\rangle \otimes |j_2, j_2\rangle$. It is annihilated by the total J_+ and has $m = j_1 + j_2$, so it generates a spin- $(j_1 + j_2)$ subspace. Because the tensor product is unitary, its orthogonal complement is invariant, and the process can be repeated.

To count the resulting spins, let w_m be the number of product states with $m_1 + m_2 = m$. For nonnegative allowed m , direct counting gives

$$w_m = \begin{cases} 2j_2 + 1, & 0 \leq m \leq j_1 - j_2, \\ j_1 + j_2 - m + 1, & j_1 - j_2 \leq m \leq j_1 + j_2. \end{cases}$$

Only weights having the appropriate integer or half-integer spacing are included. Each spin- j multiplet contributes one state at every weight $-j, -j+1, \dots, j$. Therefore the number of spin- j multiplets is $w_j - w_{j+1}$. The displayed counts give one multiplet for every $j = j_1 - j_2, \dots, j_1 + j_2$ and none below this range.

Finally the dimensions agree:

$$\begin{aligned} \sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) &= (2j_2+1)[(j_1-j_2) + (j_1+j_2) + 1] \\ &= (2j_1+1)(2j_2+1). \end{aligned}$$

Thus no states remain outside the stated decomposition. $\square$

The change from product states to simultaneous total- J^2, J_3 eigenstates is written

$$|j, m\rangle = \sum_{m_1, m_2} C_{j_1 m_1, j_2 m_2}^{j m} |j_1 m_1\rangle \otimes |j_2 m_2\rangle.$$

The coefficients are the **Clebsch–Gordan coefficients**. They vanish unless $m = m_1 + m_2$. Since both bases are orthonormal, their coefficient matrix is unitary. Their values can be obtained by starting at a highest-weight state, lowering, normalizing, and choosing orthogonal states for the remaining multiplets.

51.5 Worked example: two spin-one-half systems

Write $|\uparrow\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ and $|\downarrow\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$. The four product states are $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$. The highest state is

$$|1, 1\rangle = |\uparrow\uparrow\rangle.$$

Apply the total lowering operator:

$$J_-|\uparrow\uparrow\rangle = |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle = \sqrt{2}|1, 0\rangle.$$

Thus

$$|1, 0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}.$$

Lowering once more gives $J_-|1, 0\rangle = \sqrt{2}|\downarrow\downarrow\rangle$, so $|1, -1\rangle = |\downarrow\downarrow\rangle$.

The remaining normalized state has $m = 0$ and is orthogonal to the triplet:

$$|0, 0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}.$$

Both J_+ and J_- annihilate it, since the contributions from the two product states cancel. Together with $J_3|0, 0\rangle = 0$, this proves that it is a spin-zero singlet. Therefore

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0.$$

The triplet states are symmetric under interchange of the two factors, while the singlet is antisymmetric.

As a useful check, the scalar product of the two spins obeys

$$\mathbf{J}^{(1)} \cdot \mathbf{J}^{(2)} = \frac{1}{2} [J^2 - (J^{(1)})^2 - (J^{(2)})^2].$$

Since each individual Casimir is $3/4$, this operator has eigenvalue $1/4$ in the triplet and $-3/4$ in the singlet.

51.6 Worked example: spin one coupled to spin one-half

The six-dimensional product decomposes as $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$. Start from

$$|\frac{3}{2}, \frac{3}{2}\rangle = |1, 1\rangle|\uparrow\rangle.$$

Lowering gives

$$\sqrt{3}|\frac{3}{2}, \frac{1}{2}\rangle = \sqrt{2}|1, 0\rangle|\uparrow\rangle + |1, 1\rangle|\downarrow\rangle,$$

so

$$|\frac{3}{2}, \frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1, 0\rangle|\uparrow\rangle + \frac{1}{\sqrt{3}}|1, 1\rangle|\downarrow\rangle.$$

The orthogonal normalized combination in the same $m = \frac{1}{2}$ space can be chosen as

$$|\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|1, 0\rangle|\uparrow\rangle - \sqrt{\frac{2}{3}}|1, 1\rangle|\downarrow\rangle.$$

Acting with J_+ gives zero: the first term contributes $\sqrt{2/3}|1, 1\rangle|\uparrow\rangle$, which cancels the second. It is therefore the highest state of the spin-one-half multiplet. The other three

states follow by lowering:

$$\begin{aligned} |\tfrac{3}{2}, -\tfrac{1}{2}\rangle &= \frac{1}{\sqrt{3}}|1, -1\rangle|\uparrow\rangle + \sqrt{\tfrac{2}{3}}|1, 0\rangle|\downarrow\rangle, \\ |\tfrac{3}{2}, -\tfrac{3}{2}\rangle &= |1, -1\rangle|\downarrow\rangle, \\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}|1, -1\rangle|\uparrow\rangle - \frac{1}{\sqrt{3}}|1, 0\rangle|\downarrow\rangle. \end{aligned}$$

This example exhibits the complete highest-weight construction rather than relying on a table of coefficients.

52 The Lorentz group and its generators

52.1 Minkowski space and metric preservation

Set $c = 1$ and write

$$x^\mu = (t, x, y, z), \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1).$$

Repeated Greek indices are summed from 0 to 3. The Minkowski scalar product is

$$x \cdot y = \eta_{\mu\nu} x^\mu y^\nu = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3.$$

Unlike a Euclidean scalar product, it is not positive definite. A nonzero vector can have negative, positive, or zero squared length.

A Lorentz transformation is a real linear map $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ preserving this product. Substitution gives

$$x' \cdot y' = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho y^\sigma,$$

so preservation for all x, y is equivalent to

$$\Lambda^\top \eta \Lambda = \eta, \quad \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}. \quad (52.1)$$

The Lorentz group is $O(1, 3)$, the set of all such matrices.

It is indeed a group. If Λ_1, Λ_2 preserve η , then

$$(\Lambda_1 \Lambda_2)^\top \eta (\Lambda_1 \Lambda_2) = \Lambda_2^\top \eta \Lambda_2 = \eta.$$

Moreover,

$$\Lambda^{-1} = \eta^{-1} \Lambda^\top \eta = \eta \Lambda^\top \eta,$$

so the inverse also preserves the metric. The identity clearly does.

52.2 The four connected components

Taking the determinant of (52.1) gives $(\det \Lambda)^2 \det \eta = \det \eta$, hence $\det \Lambda = \pm 1$. The $(0, 0)$ component gives

$$-(\Lambda^0{}_0)^2 + \sum_{i=1}^3 (\Lambda^i{}_0)^2 = -1,$$

so $|\Lambda^0_0| \geq 1$. Neither the determinant sign nor the sign of Λ^0_0 can change along a continuous path of Lorentz matrices.

A transformation is **proper** if $\det \Lambda = 1$ and **orthochronous** if $\Lambda^0_0 \geq 1$. The identity component is the proper orthochronous subgroup

$$\text{SO}^+(1, 3) = L_+^\uparrow.$$

Define spatial parity and time reversal by

$$P = \text{diag}(1, -1, -1, -1), \quad T = \text{diag}(-1, 1, 1, 1).$$

They both preserve η , and

$$O(1, 3) = L_+^\uparrow \cup PL_+^\uparrow \cup TL_+^\uparrow \cup PTL_+^\uparrow.$$

The two signs are $(\det \Lambda, \text{sgn } \Lambda^0_0)$. For the four pieces above they are, respectively,

$$(+, +), \quad (-, +), \quad (-, -), \quad (+, -).$$

To understand connectedness of $L_+^\uparrow$, note that the first column of a matrix in this subgroup is a future-directed unit timelike vector. A boost can carry it to $(1, 0, 0, 0)^\top$. The remaining transformation fixes this vector and hence is a spatial rotation. Boosts and rotations can both be changed continuously to the identity, so every proper orthochronous transformation lies in that connected component.

52.3 Explicit rotations and boosts

A spatial rotation embeds as

$$\Lambda(R) = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \quad R \in \text{SO}(3).$$

It fixes time and preserves the spatial scalar product, so it satisfies (52.1) and lies in $L_+^\uparrow$.

For a frame moving along the x direction with velocity parameter β , the standard boost is

$$B_x(\xi) = \begin{pmatrix} \cosh \xi & -\sinh \xi & 0 & 0 \\ -\sinh \xi & \cosh \xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \tanh \xi. \quad (52.2)$$

The rapidity $\xi \in \mathbb{R}$ obeys $\gamma = (1 - \beta^2)^{-1/2} = \cosh \xi$ and $\gamma\beta = \sinh \xi$. Thus the upper block is also $\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$. Directly,

$$\begin{aligned} -t'^2 + x'^2 &= -(t \cosh \xi - x \sinh \xi)^2 + (x \cosh \xi - t \sinh \xi)^2 \\ &= (\cosh^2 \xi - \sinh^2 \xi)(-t^2 + x^2) = -t^2 + x^2. \end{aligned}$$

The transverse coordinates are unchanged. Also $\det B_x = 1$ and $(B_x)^0_0 = \cosh \xi \geq 1$.

The hyperbolic addition identities give $B_x(\xi_1)B_x(\xi_2) = B_x(\xi_1 + \xi_2)$. Therefore collinear rapidities add, while velocities obey

$$\beta_{12} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}.$$

For a unit direction $\hat{n}$, the corresponding boost is

$$B(\hat{n}, \xi) = \begin{pmatrix} \cosh \xi & -\sinh \xi \hat{n}^\top \\ -\sinh \xi \hat{n} & I_3 + (\cosh \xi - 1)\hat{n}\hat{n}^\top \end{pmatrix}.$$

Its unbounded entries as $|\xi| \rightarrow \infty$ show that the Lorentz group is noncompact.

52.4 Linearizing the Lorentz condition

Write a transformation near the identity as $\Lambda = I + \omega + O(\omega^2)$ with a small real matrix ω . Then

$$\Lambda^\top \eta \Lambda = \eta + \omega^\top \eta + \eta \omega + O(\omega^2).$$

The first-order condition is

$$\omega^\top \eta + \eta \omega = 0. \quad (52.3)$$

Define $\omega_{\mu\nu} = \eta_{\mu\rho}\omega^\rho{}_\nu$. Equation (52.3) becomes $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Thus there are $4 \cdot 3/2 = 6$ independent real parameters.

The mixed-index matrix $\omega^\mu{}_\nu$ is not an ordinary antisymmetric matrix: its time-space entries are symmetric, while its spatial block is antisymmetric. Lowering the first index with the Minkowski metric produces the antisymmetric tensor. This is the origin of the different matrix forms for boosts and rotations.

52.5 All six generator matrices

Use the convention

$$R(\boldsymbol{\theta}) = \exp(-i\theta_i J_i), \quad B(\boldsymbol{\xi}) = \exp(-i\xi_i K_i).$$

On four-vectors, the rotation generators have the spatial matrices J_i^{vec} in their lower-right blocks:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The boosts giving the negative off-diagonal entries in (52.2) are

$$K_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

All six obey $T^\top \eta + \eta T = 0$. The J_i are Hermitian, whereas the displayed K_i are anti-Hermitian in the ordinary positive-definite Euclidean inner product on components.

For instance, on the t, x block, $K_1 = -i\sigma_1$, so

$$e^{-i\xi K_1} = e^{-\xi \sigma_1} = I_2 \cosh \xi - \sigma_1 \sinh \xi.$$

This recovers the boost matrix directly from its generator.

52.6 Commutation relations and their physical content

Let E_{ab} have a 1 in row a , column b , and zero elsewhere. The identity $E_{ab}E_{cd} = \delta_{bc}E_{ad}$ makes the generator products easy to evaluate. In particular, $K_i = -i(E_{0i} + E_{i0})$, so for distinct i, j ,

$$[K_i, K_j] = -(E_{ij} - E_{ji}).$$

Since $J_3 = -iE_{12} + iE_{21}$, this gives $[K_1, K_2] = -iJ_3$. The same calculation for rotations and mixed products gives the full Lorentz algebra:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k, \\ [J_i, K_j] &= i\epsilon_{ijk}K_k, \\ [K_i, K_j] &= -i\epsilon_{ijk}J_k. \end{aligned} \tag{52.4}$$

The second relation says that the three boost generators transform as a vector under rotations. The third says that boosts in different directions do not commute; their small group commutator produces a rotation. The minus sign distinguishes the Lorentz algebra from the compact rotation algebra in four Euclidean dimensions.

For a covariant notation consistent with these matrices, define

$$(M_{\mu\nu})^\rho{}_\sigma = -i(\delta_\mu^\rho \eta_{\nu\sigma} - \delta_\nu^\rho \eta_{\mu\sigma}), \quad M_{\mu\nu} = -M_{\nu\mu}.$$

Then $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ and $K_i = M_{0i}$. Multiplying the matrix units yields

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} \\ &\quad - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}). \end{aligned} \tag{52.5}$$

For example, $[M_{01}, M_{02}] = -iM_{12}$ and $[M_{12}, M_{23}] = iM_{31}$ reproduce the boost and rotation relations. Using raised indices or reversing all generator signs changes the appearance of this formula; the explicit convention above fixes all signs here.

53 Finite-dimensional Lorentz representations

53.1 Two commuting angular-momentum algebras

Allow complex linear combinations of the six Lorentz generators and define

$$A_i = \frac{1}{2}(J_i + iK_i), \quad B_i = \frac{1}{2}(J_i - iK_i).$$

Using (52.4), calculate

$$\begin{aligned} [A_i, A_j] &= \frac{1}{4}([J_i, J_j] + i[J_i, K_j] + i[K_i, J_j] - [K_i, K_j]) \\ &= \frac{1}{4}(2i\epsilon_{ijk}J_k - 2\epsilon_{ijk}K_k) = i\epsilon_{ijk}A_k. \end{aligned}$$

Here $[K_i, J_j] = i\epsilon_{ijk}K_k$, obtained by exchanging the indices in the mixed commutator. Similarly,

$$\begin{aligned} [B_i, B_j] &= \frac{1}{4}(2i\epsilon_{ijk}J_k + 2\epsilon_{ijk}K_k) = i\epsilon_{ijk}B_k, \\ [A_i, B_j] &= \frac{1}{4}([J_i, J_j] - i[J_i, K_j] + i[K_i, J_j] + [K_i, K_j]) = 0. \end{aligned}$$

Thus the two sets independently obey angular-momentum commutation relations and commute with each other.

The precise complex-algebra statement is

$$\mathfrak{so}(1, 3) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}). \quad (53.1)$$

Each factor is the complexification of $\mathfrak{su}(2)$. The real Lorentz algebra itself is not a direct sum of two compact real $\mathfrak{su}(2)$ algebras. The factors of i in the definitions of A_i, B_i require complexification. For classifying finite-dimensional complex representations, however, the familiar spin construction applies separately to each factor.

53.2 Tensor-product irreducible representations

Choose a spin j_+ for the A algebra and a spin j_- for the B algebra. On

$$V_{(j_+, j_-)} = V_{j_+} \otimes V_{j_-},$$

let

$$A_i = J_i^{(+)} \otimes I, \quad B_i = I \otimes J_i^{(-)}.$$

The two actions commute because they act on different tensor factors. They have Casimirs

$$A^2 = j_+(j_+ + 1)I, \quad B^2 = j_-(j_- + 1)I,$$

and a basis $|j_+, m_+\rangle \otimes |j_-, m_-\rangle$, so

$$\dim V_{(j_+, j_-)} = (2j_+ + 1)(2j_- + 1). \quad (53.2)$$

This is irreducible under the *full* pair of commuting algebras. To see this, the joint eigenvalues of A_3, B_3 distinguish all basis states. Polynomials in these commuting operators can isolate any nonzero component of a vector in an invariant subspace. Independent $A_\pm$ and $B_\pm$ ladders then generate every basis state. Thus a nonzero invariant subspace must be the entire tensor product. Conversely, simultaneous highest-weight construction for the two commuting algebras gives precisely these pairs of spins.

Solving for the original generators gives

$$\begin{aligned} J_i &= J_i^{(+)} \otimes I + I \otimes J_i^{(-)}, \\ K_i &= -i(J_i^{(+)} \otimes I - I \otimes J_i^{(-)}). \end{aligned} \tag{53.3}$$

These equations allow all finite rotation and boost matrices to be computed by exponentiation.

Two Lorentz Casimirs can be expressed in terms of the spin labels:

$$\begin{aligned} J^2 - K^2 &= 2(A^2 + B^2) \\ &= 2[j_+(j_+ + 1) + j_-(j_- + 1)]I, \\ \mathbf{J} \cdot \mathbf{K} &= -i(A^2 - B^2) \\ &= -i[j_+(j_+ + 1) - j_-(j_- + 1)]I. \end{aligned}$$

The cross terms cancel because all A_i commute with all B_j .

53.3 Rotation content and the covering group

Under spatial rotations alone, the total generators in (53.3) add the two spins. Therefore

$$(j_+, j_-)|_{\text{rotations}} \cong \bigoplus_{j=|j_+-j_-|}^{j_++j_-} j.$$

An irreducible Lorentz representation can consequently be reducible as a representation of the rotation subgroup.

The connected Lorentz group has the double cover $\text{SL}(2, \mathbb{C})$. A convenient construction uses the Hermitian matrix

$$X = x^0 I_2 + x^i \sigma_i = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}.$$

Its determinant is

$$\det X = (x^0)^2 - |\mathbf{x}|^2 = -\eta_{\mu\nu} x^\mu x^\nu.$$

For $S \in \text{SL}(2, \mathbb{C})$, the transformation

$$X \mapsto X' = SXS^\dagger \tag{53.4}$$

preserves Hermiticity and the determinant. It therefore induces a real Lorentz transformation. Rotations arise from $S = e^{-i\theta \hat{n} \cdot \sigma/2}$ and boosts from $S = e^{-\xi \hat{n} \cdot \sigma/2}$. The kernel is $\{\pm I_2\}$, and these rotations and boosts generate $\text{SO}^+(1, 3)$.

The representation labeled (j_+, j_-) always gives a representation of this covering group. Its central element $-I_2$ acts as $(-1)^{2(j_++j_-)}$. It descends to a single-valued representation of $\text{SO}^+(1, 3)$ if and only if $j_+ + j_-$ is an integer. Spinor representations instead retain the nontrivial sign of the double cover.

53.4 Scalar and Weyl spinor representations

For $(j_+, j_-) = (0, 0)$, the space is one-dimensional and all generators vanish. A scalar field therefore transforms only through its spacetime argument:

$$\phi'(x) = \phi(\Lambda^{-1}x).$$

There is no component index to mix.

For the left-handed Weyl representation $(\frac{1}{2}, 0)$,

$$A_i = \frac{\sigma_i}{2}, \quad B_i = 0, \quad J_i = \frac{\sigma_i}{2}, \quad K_i = -i\frac{\sigma_i}{2}.$$

It acts on a two-component column ψ_L . Finite rotations and boosts are

$$D_L(R) = e^{-i\theta\hat{n}\cdot\sigma/2}, \quad D_L(B) = e^{-\xi\hat{n}\cdot\sigma/2}.$$

For a boost along the third axis,

$$D_L(B_z) = \begin{pmatrix} e^{-\xi/2} & 0 \\ 0 & e^{\xi/2} \end{pmatrix}.$$

The unequal real factors show explicitly that this boost matrix is not unitary in the positive-definite two-component inner product.

For the right-handed Weyl representation $(0, \frac{1}{2})$,

$$A_i = 0, \quad B_i = \frac{\sigma_i}{2}, \quad J_i = \frac{\sigma_i}{2}, \quad K_i = +i\frac{\sigma_i}{2}.$$

Rotations are represented by the same spin-one-half matrices, but boosts have the opposite sign:

$$D_R(B) = e^{+\xi\hat{n}\cdot\sigma/2}, \quad D_R(B_z) = \begin{pmatrix} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{pmatrix}.$$

More generally, if $D_L(S) = S$ for $S \in \text{SL}(2, \mathbb{C})$, then $D_R(S) = (S^\dagger)^{-1}$. These obey the same rotation law but are inequivalent representations of the full connected Lorentz group.

For either type of field, the complete transformation includes both the component matrix and the argument:

$$\psi'_L(x) = D_L(\Lambda)\psi_L(\Lambda^{-1}x), \quad \psi'_R(x) = D_R(\Lambda)\psi_R(\Lambda^{-1}x).$$

The notation $D(\Lambda)$ for a spinor includes a choice of lift to the double cover.

53.5 The Dirac direct sum and parity

A Dirac spinor combines the two Weyl representations:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad D_D = (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2}).$$

Its generators are the 4×4 block matrices

$$J_i^D = \begin{pmatrix} \sigma_i/2 & 0 \\ 0 & \sigma_i/2 \end{pmatrix}, \quad K_i^D = \begin{pmatrix} -i\sigma_i/2 & 0 \\ 0 & +i\sigma_i/2 \end{pmatrix}.$$

Each Weyl subspace is invariant under the connected Lorentz group, so this four-dimensional representation is reducible.

Parity leaves angular momentum unchanged but reverses boosts: $J_i \mapsto J_i$ and $K_i \mapsto -K_i$. Hence it exchanges A_i with B_i and maps (j_+, j_-) to (j_-, j_+) . On the Dirac direct sum, an operator proportional to

$$\mathcal{P} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

exchanges the Weyl components and has precisely these conjugation properties. A single chiral Weyl subspace is therefore not closed under parity, while the direct sum contains both chiralities.

53.6 The four-vector representation

For $(j_+, j_-) = (\tfrac{1}{2}, \tfrac{1}{2})$,

$$\dim V_{(1/2, 1/2)} = 2 \cdot 2 = 4.$$

Equation (53.4) supplies a concrete realization: the four coefficients of a Hermitian 2×2 matrix X transform as a four-vector. For example, with $S = \text{diag}(e^{-\xi/2}, e^{\xi/2})$,

$$X'_{11} = e^{-\xi}(x^0 + x^3), \quad X'_{22} = e^{\xi}(x^0 - x^3), \quad X'_{12} = X_{12}.$$

Adding and subtracting the diagonal entries yields

$$x'^0 = x^0 \cosh \xi - x^3 \sinh \xi, \quad x'^3 = x^3 \cosh \xi - x^0 \sinh \xi,$$

with $x'^1 = x^1$ and $x'^2 = x^2$. This is exactly the usual boost of a four-vector. A complex four-vector is obtained by complexifying this real four-dimensional space.

Under rotations,

$$(\tfrac{1}{2}, \tfrac{1}{2})|_{\text{rotations}} = 0 \oplus 1.$$

The scalar piece is the time component x^0 , while the spin-one piece is the spatial vector $\mathbf{x}$. Boosts mix the two pieces, so neither is an invariant subspace of the full Lorentz representation. This explains why the four-vector representation is Lorentz-irreducible even though its rotation content splits.

A vector field, such as a four-current, transforms as

$$V'^{\mu}(x) = \Lambda^{\mu}_{\nu} V^{\nu}(\Lambda^{-1}x).$$

The four-component Dirac spinor and the four-vector have equal dimension, but their generator matrices and transformation laws are different. Dimension alone does not identify a representation.

The labels (j_+, j_-) classify finite-component Lorentz field representations. The Hilbert-space representations used to classify relativistic particle states also involve spacetime translations and the Poincaré group. This distinction keeps the component transformation law separate from the state-space action of spacetime symmetries.

A A compact method reference

A.1 Choosing a method

Problem	Main steps
Real rational integral	Continue to z ; choose a contour; bound the added arc; sum enclosed residues.
Oscillatory integral	Choose the half-plane where the exponential decays; track orientation and any real-axis prescription.
Fractional powers or logarithms	Fix a branch; specify the cut; account for the different values on its two sides.
Eigenfunction expansion	Specify interval, measure, domain, and boundary conditions; solve and normalize the eigenproblem.
Inhomogeneous boundary ODE	Check zero modes and compatibility; construct G with matching and the correct derivative jump.
Causal response	Find the impulse kernel; impose $G(t < 0) = 0$; locate frequency poles and use convolution.
Translation-invariant PDE	Transform in uniform directions; solve the resulting algebraic equation or ODE; invert.
First-order PDE	Integrate characteristic ODEs; transport data along them; check transversality and possible crossing.
Bounded-domain PDE	Subtract inhomogeneous boundary data; separate variables; project data onto the eigenbasis.
Spherical potential	Expand boundary data or sources in $Y_{\ell m}$; choose radial powers using regularity and decay.
Symmetry problem	Identify the group and representation; decompose into irreducibles; use generators and invariant quantities.

B Source-to-topic index

The following index records where the supplied material is incorporated. Overlapping discussions appear together under their mathematical topic. Figures are cropped from the supplied PDFs; the accompanying `figure_sources.json` records each image's source page.

Supplied file	Main location
P605_F25(1).pdf	Introduction and reading list
Complex analysis-1.pdf	Arithmetic and analyticity
Complex-analysis-2(1).pdf	Cauchy–Riemann and Cauchy formula
Complex-analysis-3(1).pdf	Cauchy consequences and series
Complex analysis-4(1).pdf	Singularities and residues
Complex-analysis-5(1).pdf	Contours and branch cuts
Complex-analysis-6(1).pdf	Principal values and linear response
Function-spaces-1(1).pdf	Hilbert spaces and convergence
Function-spaces-2(1).pdf	Polynomials and distributions
Diff-operators-1(1).pdf	Formal adjoints and spectra
Diff-operators-2(1).pdf	Domains and self-adjointness
Diff-operators-3(1).pdf	Spectra and density of states
Green's function-1.pdf	Fredholm alternative and Green kernels
Green's function-2.pdf	Diffusion and capillary surfaces
Green (1).pdf	Boundary kernels, similarity, propagation
greens_function_EXAMPLES(1).pdf	Oscillators and causal waves
Fourier-Transform-important-Properties(1).pdf	Fourier and Hankel transforms
SphericalHarmonics.pdf	Spherical harmonics and multipoles
PDE-1(2).pdf	PDE classification and data
PDE-2(2).pdf	Characteristics and conservation
PDE-3(2).pdf	Waves and radiation damping
PDE Anazt(2).pdf	Heat-kernel similarity calculation
ODEs_ First order linear PDE.pdf	Characteristic examples and practice
conservationdissipation(1).pdf	Conservation and energy functionals
Wave-eq-characteristics(1).pdf	Waves, characteristics, polarization
Groups and representations in Physics(2).pdf	Supplementary group theory
Groups and representations- Lecture4(2).pdf	Lorentz group and representations